

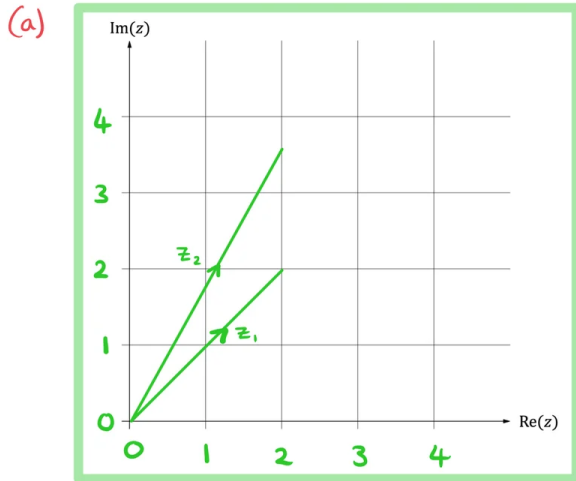
Complex Numbers

Mark Schemes

Question 1

Consider the complex numbers $z_1 = 2 + 2i$ and $z_2 = 2 + 2\sqrt{3}i$.

(a) Sketch z_1 and z_2 on the Argand diagram below, be sure to include an appropriate scale.



[2]

(b) Find the modulus of z_1 and z_2 .

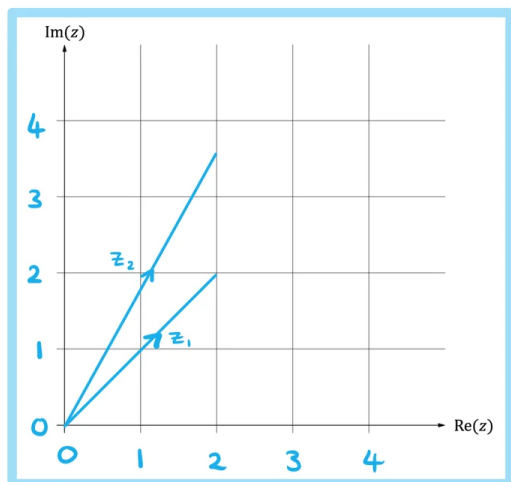
[3]

(c) Find the argument of z_1 and z_2 .

[3]

Consider the complex numbers $z_1 = 2 + 2i$ and $z_2 = 2 + 2\sqrt{3}i$.

(a) Sketch z_1 and z_2 on the Argand diagram below, be sure to include an appropriate scale.



[2]

(b) Find the modulus of z_1 and z_2 .

[3]

(c) Find the argument of z_1 and z_2 .

[3]

$$(b) |z_1| = \sqrt{2^2 + 2^2} = \sqrt{8}$$

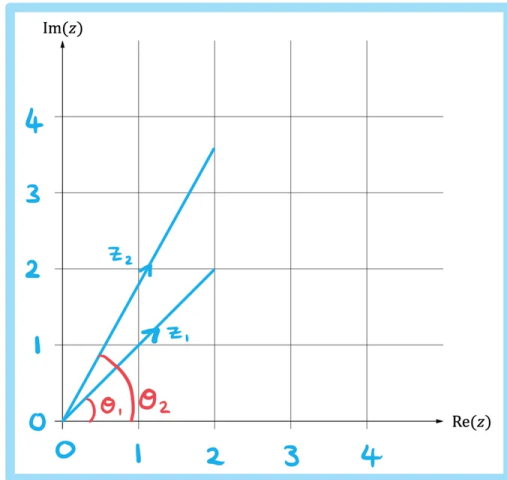
$$|z_1| = 2\sqrt{2}$$

$$|z_2| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16}$$

$$|z_2| = 4$$

Consider the complex numbers $z_1 = 2 + 2i$ and $z_2 = 2 + 2\sqrt{3}i$.

(a) Sketch z_1 and z_2 on the Argand diagram below, be sure to include an appropriate scale.



(b) Find the modulus of z_1 and z_2 .

(c) Find the argument of z_1 and z_2 .

(c) $\tan \theta_1 = \frac{2}{2} = 1$ \leftarrow Trig exact values $\tan \frac{\pi}{4} = 1$

$$\theta_1 = \frac{\pi}{4}$$



$\tan \theta_2 = \frac{2\sqrt{3}}{2} = \sqrt{3}$ \leftarrow Trig exact values $\tan \frac{\pi}{3} = \sqrt{3}$

$$\theta_2 = \frac{\pi}{3}$$



[2]

[3]

[3]

Question 2

Solve the following equations for x

(i) $x^2 + 4x + 5 = 0$

(ii) $x^2 = -625$

(iii) $x^4 = 24 - 2x^2$

$$(i) \quad x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{-1}}{2}$$

$$x = -2 \pm i$$

(ii) $x = \sqrt{-625}$

$$= \pm 25\sqrt{-1}$$

$$x = \pm 25i$$

[7]

(iii) $x^4 + 2x^2 - 24 = 0$ \leftarrow Treat like a quadratic $(x^2)^2 + 2(x^2) - 24 = 0$

$$(x^2 - 4)(x^2 + 6) = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

$$x^2 = -6$$

$$x = \pm \sqrt{-6}$$

$$x = \pm \sqrt{6}i$$

Question 3

Let $w_1 = z_1 z_2$, where $z_1 = 5 + i$ and $z_2 = 1 + 2i$.

(a) Express w in the form $w = a + bi$.

(b) Find the modulus and argument for w .

[2]

[4]

$$\begin{aligned}
 \text{(a) } w &= (5+i)(1+2i) \\
 &= 5+i+10i+2i^2 \\
 &= 5+11i-2
 \end{aligned}$$

$$w = 3 + 11i$$

Let $w_1 = z_1 z_2$, where $z_1 = 5 + i$ and $z_2 = 1 + 2i$.

(a) Express w in the form $w = a + bi$.

$$w = 3 + 11i$$

(b) Find the modulus and argument for w .

[2]

[4]

$$\text{(b) } |w| = \sqrt{3^2 + 11^2}$$

$$|w| = \sqrt{130}$$

$$\tan \theta_w = \frac{11}{3}$$

$$\theta_w = 1.30 \text{ rad}$$

Question 4

Let $z = \frac{w_1}{w_2}$, where $w_1 = 4 - i$ and $w_2 = 1 - 2i$.

(a) Express z in the form $z = a + bi$.

(b) Find the modulus and argument for z .

[3]

[4]

$$\text{(a) } z = \frac{4-i}{1-2i} \times \frac{1+2i}{1+2i} \quad \leftarrow \text{Multiply by the conjugate to cancel out } i \text{ in the denominator}$$

$$= \frac{4-i+8i-2i^2}{1-2i+2i-4i^2}$$

$$= \frac{4+7i+2}{1+4}$$

$$= \frac{6+7i}{5}$$

$$z = \frac{6}{5} + \frac{7i}{5}$$

Let $z = \frac{w_1}{w_2}$, where $w_1 = 4 - i$ and $w_2 = 1 - 2i$.

(a) Express z in the form $z = a + bi$.

$$z = \frac{6}{5} + \frac{7i}{5}$$

(b) Find the modulus and argument for z .

[3]

$$(b) |z| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{7}{5}\right)^2}$$

$$|z| = \frac{\sqrt{85}}{5}$$

[4]

$$\tan \theta_z = \frac{\left(\frac{7}{5}\right)}{\left(\frac{6}{5}\right)}$$

$$\theta_z = 0.862 \text{ rad}$$

Question 5

Consider the complex numbers $z = 3 - 4i$ and $w = 7 - 2i$.

(a) Find

(i) $z + w$

(ii) $w - z$

Let z^* and w^* represent the complex conjugates of z and w , respectively.

(b) Write down z^* and w^* , giving your answers in the form $a + bi$.

(c) Find

(i) z^*w

(ii) $\frac{w^*}{z}$

$$(a) (i) z + w = (3 - 4i) + (7 - 2i)$$

$$z + w = 10 - 6i$$

[2]

$$(ii) w - z = (7 - 2i) - (3 - 4i)$$

$$w - z = 4 + 2i$$

[2]

[4]

Consider the complex numbers $z = 3 - 4i$ and $w = 7 - 2i$.

(a) Find

(i) $z + w$

(ii) $w - z$.

Let z^* and w^* represent the complex conjugates of z and w , respectively.

(b) Write down z^* and w^* , giving your answers in the form $a + bi$.

(c) Find

(i) z^*w

(ii) $\frac{w^*}{z}$.

[2]

[2]

[4]

(b)
$$\begin{aligned} z^* &= 3 + 4i \\ w^* &= 7 + 2i \end{aligned}$$

Consider the complex numbers $z = 3 - 4i$ and $w = 7 - 2i$.

(a) Find

(i) $z + w$

(ii) $w - z$.

Let z^* and w^* represent the complex conjugates of z and w , respectively.

(b) Write down z^* and w^* , giving your answers in the form $a + bi$.

(c) Find

(i) z^*w

(ii) $\frac{w^*}{z}$

$$\begin{aligned} z^* &= 3 + 4i \\ w^* &= 7 + 2i \end{aligned}$$

[2]

[2]

[4]

(c) (i)
$$\begin{aligned} z^*w &= (3 + 4i)(7 - 2i) \\ &= 21 + 28i - 6i + 8 \end{aligned}$$

$$z^*w = 29 + 22i$$

(ii)
$$\begin{aligned} \frac{w^*}{z} &= \frac{7 + 2i}{3 - 4i} \times \frac{3 + 4i}{3 + 4i} \\ &= \frac{21 + 6i + 28i - 8}{9 - 12i + 12i + 16} \\ &= \frac{13 + 34i}{25} \end{aligned}$$

← Multiply by the conjugate to cancel out i in the denominator

$$\frac{w^*}{z} = \frac{13}{25} + \frac{34i}{25}$$

Question 6

Find all possible real values for a and b such that

- (i) $a + bi = 8i$
- (ii) $(2 + 3i)(a + bi) = 13$
- (iii) $(a + i)(2 + bi) = -6 + 22i$

(i) $a = 0 \quad b = 8$

(ii) $(2 + 3i)(a + bi) = 13$
 $2a + 3ai + 2bi - 3b = 13$
 $2a + 3ai + 2bi - 3b = 13$
 $(2a - 3b) + (3a + 2b)i = 13$
 Real parts must sum to 13
 Imaginary parts must sum to 0
 $2a - 3b = 13$ $3a + 2b = 0$

(iii) $(a + i)(2 + bi) = -6 + 22i$
 $2a + 2i + abi - b = -6 + 22i$
 $(2a - b) + (2 + ab)i = -6 + 22i$

Equate the real parts and the imaginary parts

$$2a - b = -6 \rightarrow b = 2a + 6$$

$$2 + ab = 22 \rightarrow ab = 20$$

Solve equations simultaneously to find a and b

$$a(2a + 6) = 20$$

$$2a^2 + 6a - 20 = 0$$

$$a^2 + 3a - 10 = 0$$

$$(a + 5)(a - 2) = 0$$

$a = -5$ $a = 2$
 $b = 2(-5) + 6 = -4$ $b = 2(2) + 6 = 10$

$a = -5 \quad b = -4$ or $a = 2 \quad b = 10$

Solve equations simultaneously to find a and b

$$2a - 3b = 13 \rightarrow 6a - 9b = 39$$

$$3a + 2b = 0 \rightarrow 6a + 4b = 0$$

$$-13b = 39$$

$b = -3$

$$3a + 2(-3) = 0$$

$$3a - 6 = 0$$

$a = 2$

[7]

Question 7

Consider the complex numbers $w = iz$ and $w + 2z = 7 + 6i$.

Find

- (i) $\text{Re}(w)$
- (ii) $\text{Im}(w)$
- (iii) $\text{Re}(z)$
- (iv) $\text{Im}(z)$.

Write z and w in terms of two unknown variables x and y

$$z = x + yi \rightarrow w = i(x + yi)$$

Replace z and w in the equation $w + 2z = 7 + 6i$ to find the values of x and y

$$i(x + yi) + 2(x + yi) = 7 + 6i$$

$$xi - y + 2x + 2yi = 7 + 6i$$

$$(2x - y) + (x + 2y)i = 7 + 6i$$

Equating the real parts and the imaginary parts from both sides, then solve simultaneously to find x and y

$$\begin{array}{r} 2x - y = 7 \\ x + 2y = 6 \rightarrow 2x + 4y = 12 \\ \hline -5y = -5 \\ y = 1 \end{array}$$

$$\begin{array}{l} x + 2(1) = 6 \\ x = 4 \end{array}$$

Substitute the values for x and y back into the equations for w and z

$$z = 4 + i$$

$$w = i(4 + i) = -1 + 4i$$

(i) $\text{Re}(w) = -1$

(ii) $\text{Im}(w) = 4i$

(iii) $\text{Re}(z) = 4$

(iv) $\text{Im}(z) = i$

[7]

Question 8

It is given that $z_1 = 3 + 4i$ and $z_2 = -2 + 2i$.

Find

- (i) $iz_1 + z_2$
- (ii) $\frac{z_1}{iz_2}$
- (iii) $i(z_1 z_2)$.

(i) $iz_1 + z_2 = i(3 + 4i) + (-2 + 2i)$
 $= 3i - 4 - 2 + 2i$

$$iz_1 + z_2 = -6 + 5i$$

(ii) $\frac{z_1}{iz_2} = \frac{3 + 4i}{i(-2 + 2i)}$

$$= \frac{3 + 4i}{-2 - 2i}$$

$$= \frac{3 + 4i}{-2 - 2i} \times \frac{-2 + 2i}{-2 + 2i}$$

$$= \frac{-6 - 8i + 6i - 8}{4 + 2i - 2i + 4}$$

$$= \frac{-14 - 2i}{8}$$

← Multiply by the conjugate to cancel out i in the denominator

$$\frac{z_1}{iz_2} = -\frac{7}{4} - \frac{1}{4}i$$

[7]

(iii) $i(z_1 z_2) = i(3 + 4i)(-2 + 2i)$

$$= i(-6 - 8i + 6i - 8)$$

$$= -6i + 8 - 6 - 8i$$

$$i(z_1 z_2) = 2 - 14i$$

Question 9

Find the complex numbers z and w such that

$$2z - iw^* = 5 + 7i$$

$$w + iz^* = 5 + 16i$$

[8]

Write z in terms of two unknown variables x and y and write w in terms of two unknown variables a and b

$$z = x + yi \quad w = a + bi$$

Replace z and w and their conjugates in to both equations and equate the real parts and the imaginary parts

$$1^{st} \text{ equation: } 2(x + yi) - i(a - bi) = 5 + 7i$$

$$2x + 2yi - ai - b = 5 + 7i$$

$$(2x - b) + (2y - a)i = 5 + 7i$$

$$\Rightarrow 2x - b = 5, \quad 2y - a = 7$$

$$2^{nd} \text{ equation: } (a + bi) + i(x - yi) = 5 + 16i$$

$$a + bi + xi + y = 5 + 16i$$

$$(y + a) + (x + b)i = 5 + 16i$$

$$\Rightarrow y + a = 5, \quad x + b = 16$$

Solve both pairs of simultaneous equations for x, y, a and b

$$2x - b = 5$$

$$x + b = 16 \quad +$$

$$3x = 21$$

$$x = 7$$

$$(7) + b = 16$$

$$b = 9$$

$$2y - a = 7$$

$$y + a = 5 \quad +$$

$$3y = 12$$

$$y = 4$$

$$(4) + a = 5$$

$$a = 1$$

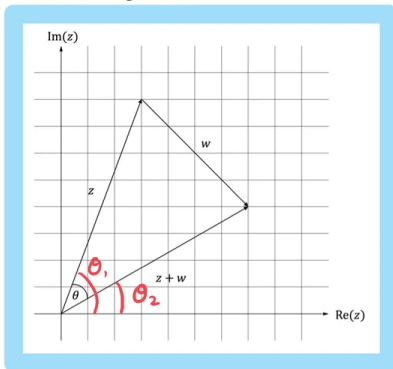
$$z = 7 + 4i$$

$$w = 1 + 9i$$

Question 10

Let $z = 3 + 8i$ and $w = 4 - 4i$.

(a) Find θ , the angle shown on the diagram below.



(b) Find the area of the triangle formed in the diagram above.

(a) Find the argument of z , θ_1

$$\tan \theta_1 = \frac{8}{3}$$

$$\theta_1 = 1.21202565\dots$$

Find $z + w$ and its argument, θ_2

$$z + w = 7 + 4i$$

$$\tan \theta_2 = \frac{4}{7}$$

$$\theta_2 = 0.519146114\dots$$

[5]

Find θ

[3]

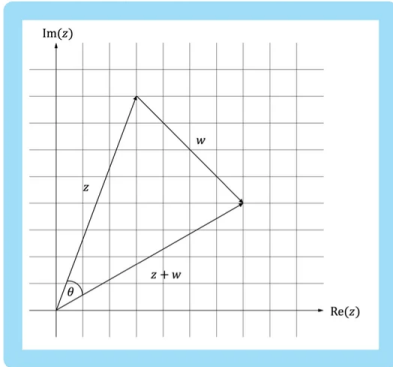
$$\theta_1 - \theta_2 = 1.21202565\dots - 0.519146114\dots$$

$$\theta = 0.693 \text{ rad}$$

Let $z = 3 + 8i$ and $w = 4 - 4i$.

$z + w = 7 + 4i$

(a) Find θ , the angle shown on the diagram below.



(b) Find the area of the triangle formed in the diagram above.

$\theta = 0.693 \text{ rad}$

(b) Find $|z|$ and $|z+w|$

$$|z| = \sqrt{3^2 + 8^2}$$

$$= \sqrt{73}$$

$$|z+w| = \sqrt{7^2 + 4^2}$$

$$= \sqrt{65}$$

$$A = \frac{1}{2}ab\sin C \quad \leftarrow \text{Formula booklet}$$

[5]

$$A = \frac{1}{2} \sqrt{73} \sqrt{65} \sin 0.6928795\dots$$

$$= 21.9999999\dots$$

[3]

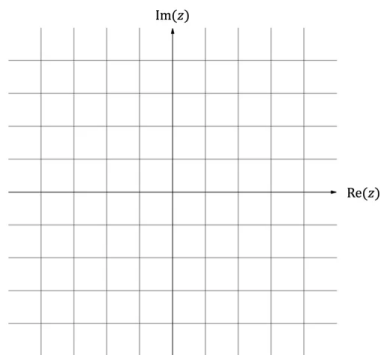
$A = 22.0 \text{ units}^2$

Question 11

Let $z = -1 - 3i$ and $w = 1 + i$.

(a) Find zw .

(b) Sketch z , w and zw on the Argand diagram below.



(a) $zw = (-1 - 3i)(1 + i)$

$$= -1 - 3i - i + 3$$

[2]

$zw = 2 - 4i$

Let θ be the angle between z and zw and ϕ be the angle between w and zw .

(c) Find the angles θ and ϕ , giving your answers in degrees.

[3]

[4]

Let $z = -1 - 3i$ and $w = 1 + i$.

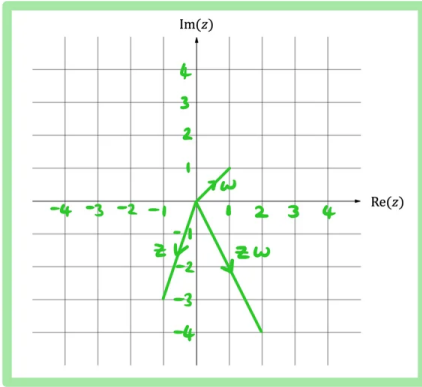
(a) Find zw .

$zw = 2 - 4i$

[2]

(b) Sketch z , w and zw on the Argand diagram below.

(b)



[3]

Let θ be the angle between z and zw and ϕ be the angle between w and zw .

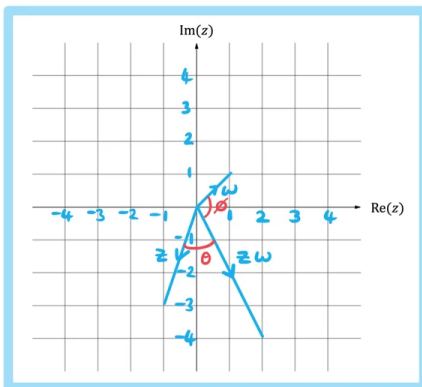
(c) Find the angles θ and ϕ , giving your answers in degrees.

[4]

Let $z = -1 - 3i$ and $w = 1 + i$.

(a) Find zw .

(b) Sketch z , w and zw on the Argand diagram below.



[3]

Let θ be the angle between z and zw and ϕ be the angle between w and zw .

(c) Find the angles θ and ϕ , giving your answers in degrees.

[4]

(c) Find θ

[2]

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{4}\right)$$

$\theta = 45^\circ$

Find ϕ

$$\phi = \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{4}{2}\right)$$

$\phi = 108^\circ$

Question 12

Let $w = \frac{z+1}{z^2+1}$, where $z = a + bi$, $a, b \in \mathbb{R}$.

(a) Write w in the form $x + yi$, $x, y \in \mathbb{R}$.

(b) Determine the conditions under which w is purely imaginary.

[4]

[3]

(a) Replace z and its conjugate in terms of a and b

$$w = \frac{(a+bi)+1}{(a-bi)+1}$$

Group the real parts together

$$= \frac{(a+1) + bi}{(a+1) - bi}$$

$$= \frac{(a+1) + bi}{(a+1) - bi} \times \frac{(a+1) + bi}{(a+1) + bi}$$

$$= \frac{(a+1)^2 + 2(a+1)bi - b^2}{(a+1)^2 + b^2}$$

Multiply by the conjugate to cancel out i in the denominator

$$w = \left(\frac{(a+1)^2 - b^2}{(a+1)^2 + b^2} \right) + \left(\frac{2(a+1)b}{(a+1)^2 + b^2} \right) i$$

Let $w = \frac{z+1}{z^2+1}$, where $z = a + bi$, $a, b \in \mathbb{R}$.

(a) Write w in the form $x + yi$, $x, y \in \mathbb{R}$.

$$w = \left(\frac{(a+1)^2 - b^2}{(a+1)^2 + b^2} \right) + \left(\frac{2(a+1)b}{(a+1)^2 + b^2} \right) i$$

[4]

(b) Determine the conditions under which w is purely imaginary.

[3]

(b) If w is imaginary the real part must be equal to 0

$$\frac{(a+1)^2 - b^2}{(a+1)^2 + b^2} = 0$$

$$(a+1)^2 - b^2 = 0$$

$$(a+1)^2 = b^2$$

$$\pm(a+1) = b$$

So b must be 1 more than a if a is positive and 1 less than a if a is negative

Looking at the real part of the original equation, you can see that $a \neq -1$ or $b = 0$

$$\pm(a+1) = b \quad \text{and} \quad a \neq -1, \quad b \neq 0$$

Question 13

Consider the equation $x^2 + bx + c = 0$.

(a) Write down an inequality, in terms of b and c , that shows the equation has no real solutions.

[2]

$5 - 3i$ is one solution to the equation $x^2 + bx + c = 0$.

(b) Find the values of b and c .

[4]

Let $z = c + bi$.

(c) Find z^5 using technology.

[1]

(a) The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$ ↪ formula booklet

For there to be no real solutions, the discriminant must be less than 0

$$b^2 - 4ac < 0$$

$$b^2 - 4c < 0$$

Consider the equation $x^2 + bx + c = 0$.

(a) Write down an inequality, in terms of b and c , that shows the equation has no real solutions.

[2]

$5 - 3i$ is one solution to the equation $x^2 + bx + c = 0$.

(b) Find the values of b and c .

[4]

Let $z = c + bi$.

(c) Find z^5 using technology.

[1]

(b) The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$ ↪ formula booklet

$$\frac{-b - \sqrt{b^2 - 4c}}{2} = 5 - 3i$$

$$-b - \sqrt{b^2 - 4c} = 10 - 6i$$

Compare the real parts from both sides

$$b = -10$$

Substitute $b = -10$ into equation and compare the imaginary parts from both sides

$$-\sqrt{(-10)^2 - 4c} = -6i$$

$$100 - 4c = 36i^2$$

$$136 = 4c$$

$$c = 34$$

Consider the equation $x^2 + bx + c = 0$.

(a) Write down an inequality, in terms of b and c , that shows the equation has no real solutions.

[2]

$5 - 3i$ is one solution to the equation $x^2 + bx + c = 0$.

(b) Find the values of b and c .

$$b = -10$$

$$c = 34$$

[4]

Let $z = c + bi$.

(c) Find z^5 using technology.

[1]

(c) $z^5 = 7831424 - 55356800i$