Question 1

Find the coefficient of the term in x^3 in the expansion of $(2-x)^8$.

[3]

$n = 8, \quad a = 2, \quad b = -x$

Substitute values into the formula for the binomial theorem:

$(a+b)^n = a^n + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$

where ${}^nC_r = \frac{n!}{r!(n-r)!}$

$(2-x)^8 = \sum_{r=0}^{8} {}^8C_r\, (2)^{8-r}\, (-x)^r$ ← Coefficient of x^3 occurs when $r = 3$.

$r = 3$ gives ${}^8C_3 \times 2^{8-3}\,(-x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^8C_3 = \frac{8!}{3!(8-3)!} = \frac{8\times7\times6\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}}{(3\times2)(\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2})}$$

$$= \frac{8\times7\times\cancel{6}}{\cancel{6}} = 56$$

so the term when $r=3$ is $56 \times 2^5 \times (-x)^3$

$= 56 \times -32x^3$

$= -1792\,x^3$

$$\boxed{\text{Coefficient of } x^3 = -1792}$$

Question 2

Find the first three terms, in ascending powers of x, in the expansion of $(3+x)^4$.

[3]

$a = 3, \quad b = x, \quad n = 4$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

Question asks for ascending powers of x, so start with the term in x^0.

$(3+x)^4 = \underbrace{3^4}_{\text{constant term}} + \underbrace{{}^4C_1 (3)^{4-1}(x)^1}_{\text{term in } x} + \underbrace{{}^4C_2 (3)^{4-2}(x)^2}_{\text{term in } x^2} + \ldots$

$\approx 81 + \dfrac{4!}{3!} \times 3^3 \times x + \dfrac{4!}{2!2!} \times 3^2 \times x^2$

$\approx 81 + 4 \times 27x + 6 \times 9x^2$

$\approx 81 + 108x + 54x^2$

$$\boxed{(3+x)^4 \approx 81 + 108x + 54x^2}$$

Question 3

In the expansion of $(a-x)^4$, the coefficient of the x^2 term is 96.
Given that $a > 0$, find the value of a.

[4]

$a = a, \quad b = -x, \quad n = 4$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = \sum_{r=0}^{n} {}^nC_r \, a^{n-r} b^r$

$(a-x)^4 = \sum_{r=0}^{4} {}^4C_r \, a^{4-r} (-x)^r$ given the coefficient of the term in x^2, so evaluate the term when $r=2$.

Term in $x^2 = {}^4C_2 (a)^{4-2} (-x)^2 = 96x^2$

$6a^2(-x)^2 = 96x^2$

coefficient of x^2

$6a^2 = 96$

$a^2 = 16$

$a = \pm 4$

It is given in the question that $a>0$ $\Rightarrow a = 4$

$\boxed{a = 4}$

Question 4

Find the first three terms, in ascending powers of x, in the expansion of $(9-2x)^5$.

[3]

$a = 9 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r \, a^{n-r} b^r + \ldots + b^n$

Question asks for ascending powers of x, so start with the constant term, a^n.

$(9-2x)^5 = 9^5 + {}^5C_1 (9)^{5-1}(-2x) + {}^5C_2 (9)^{5-2}(-2x)^2 + \ldots$

$\approx 59049 + 5 \times 6561 \times -2x + 10 \times 729 \times 4x^2$

$\approx 59049 - 65610x + 29160x^2$

$\boxed{(9-2x)^5 \approx 59049 - 65610x + 29160x^2}$

Question 5

In the expansion of $(a-2x)^5$, the coefficient of the x^2 term is equal to the coefficient of the x^3 term. Find the value of a.

[4]

$a = a, \quad b = -2x, \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$$

$$(a-2x)^5 = \sum_{r=0}^{5} {}^5C_r\, a^{5-r}(-2x)^r$$

The coefficients of the terms in x^2 and x^3 are equal, so evaluate the terms when $r=2$ and $r=3$.

In x^2 term, $r=2$:

$${}^5C_2\, a^{5-2}(-2x)^2$$
$$= 10\, a^3 \times 4x^2$$
$$= 40a^3x^2$$

coefficient of x^2 term

In x^3 term, $r=3$:

$${}^5C_3\, a^{5-3}(-2x)^3$$
$$= 10\, a^2 \times -8x^3$$
$$= -80a^2x^3$$

coefficient of x^3 term

Equating coefficients: $40a^3 = -80a^2$

$$a = \frac{-80}{40} = -2$$

$$\boxed{a = -2}$$

Question 6

In the expansion of $(3+px)^6$, the coefficient of the x^4 term is four times the coefficient of the x^2 term. Find the possible values of p.

[3]

$a = 3 \quad b = px \quad n = 6$

$$(3+px)^6 = \sum_{r=0}^{6} {}^6C_r\,(3)^{6-r}(px)^r$$

evaluate the terms when $r=2$ and $r=4$.

In x^2 term, $r=2$:

$${}^6C_2\,(3)^{6-2}(px)^2$$
$$= 15 \times 81 \times p^2x^2$$
$$= 1215p^2x^2$$

coefficient of x^2

In x^4 term, $r=4$:

$${}^6C_4\,(3)^{6-4}(px)^4$$
$$= 15 \times 9 \times p^4x^4$$
$$= 135p^4x^4$$

coefficient of x^4

coefficient of x^4 = 4(coefficient of x^2)

$$135p^4 = 4(1215p^2)$$
$$135p^4 = 4860p^2$$
$$p^2 = \frac{4860}{135} = 36$$
$$\Rightarrow \quad p = \sqrt{36} = \pm 6$$

$$\boxed{p = 6 \text{ or } -6}$$

Question 7

Consider the expansion of $(4ax - 3)^5$.

(a) Write down the number of terms in this expansion.

[1]

(b) The coefficient of the term in x^4 is -61440.
Find the value of a where a is a positive constant.

[4]

(a) A binomial expansion has $n+1$ terms.

$n = 5 \Rightarrow n+1 = 6$

$\boxed{(4ax-3)^5 \text{ has } 6 \text{ terms}}$

Consider the expansion of $(4ax - 3)^5$.

(a) Write down the number of terms in this expansion.

[1]

(b) The coefficient of the term in x^4 is -61440.
Find the value of a where a is a positive constant.

[4]

(b) $a = 4ax, \quad b = -3, \quad n = 5$

$(4ax - 3)^5 = \sum_{r=0}^{5} {}^5C_r (4ax)^{5-r}(-3)^r$

we have been given the coefficient of the term in x^4, so evaluate the term when $r = 1$.

Term in $x^4 = -61440x^4$

$\Rightarrow {}^5C_1(4ax)^{5-1}(-3)^1 = -61440x^4$

$5 \times -3 \times 4^4 \times a^4 \times x^4 = -61440x^4$

$-3840a^4 = -61440$

$a^4 = \frac{-61440}{-3840} = 16$

$a = \sqrt[4]{16} = \pm 2$

It is given in the question that a is a positive constant.

$\boxed{a = 2}$

Question 8

Consider the expansion of $(x^3 + \frac{4}{x})^4$.

(a) Write the first three terms in descending powers of x. [3]

(b) Find the value of the constant term. [3]

(a) $a = x^3$, $b = \frac{4}{x}$, $n = 4$

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$$

$$\left(x^3 + \frac{4}{x}\right)^4 = \sum_{r=0}^{4} {}^4C_r\, (x^3)^{4-r} \left(\frac{4}{x}\right)^r$$

The numerator has the greater power of x, so start with $r=0$ as the first term

$$= \sum_{r=0}^{4} {}^4C_r \left(\frac{4^r x^{3(4-r)}}{x^r}\right)$$

$$\left(x^3 + \frac{4}{x}\right)^4 = {}^4C_0\left(\frac{4^0 x^{3(4-0)}}{x^0}\right) + {}^4C_1\left(\frac{4^1 x^{3(4-1)}}{x^1}\right) + {}^4C_2\left(\frac{4^2 x^{3(4-2)}}{x^2}\right) + \dots$$

$$\approx \frac{1 \times 1x^{12}}{1} + 4 \times \frac{4x^9}{x} + 6 \times \frac{16x^6}{x^2}$$

$$\approx x^{12} + 16x^8 + 96x^4$$

$$\boxed{\left(x^3 + \frac{4}{x}\right)^4 \approx x^{12} + 16x^8 + 96x^4}$$

Consider the expansion of $(x^3 + \frac{4}{x})^4$.

(a) Write the first three terms in descending powers of x. [3]

(b) Find the value of the constant term. [3]

(b) $$\left(x^3 + \frac{4}{x}\right)^4 = \sum_{r=0}^{4} {}^4C_r\, (x^3)^{4-r} \left(\frac{4}{x}\right)^r$$

$$= \sum_{r=0}^{4} {}^4C_r \left(\frac{4^r x^{3(4-r)}}{x^r}\right)$$

The constant term is the term in x^0, so we need r such that $3(4-r) - r = 0$

$$3(4-r) - r = 0$$
$$12 - 4r = 0$$
$$r = 3$$

$r = 3$ gives ${}^4C_3\left(\frac{4^3\, x^{3(4-3)}}{x^3}\right) = 4 \times 4^3 \times 1$

$= 256$

$$\boxed{\text{constant term} = 256}$$

Question 9

The coefficient of x^7 in the expansion of $x^3(ax+3)^5$ is 1215.
Find the possible values of a.

[4]

First expand $(ax+3)^5$, rearranging this to $(3+ax)^5$ makes it easier to spot the correct term to use.

$$(3+ax)^5 = \sum_{r=0}^{5} {}^5C_r\,(3)^{5-r}(ax)^r$$

The question gives the term in x^7, so here we are looking for the term in x^4. Evaluate for $r=4$.

The term when $r=4$:

$$(3+ax)^5 = \ldots + {}^5C_4(3)^{5-4}(ax)^4 + \ldots$$

$$= \ldots + 15a^4x^4 + \ldots$$

The coefficient of x^7 in the expansion $x^3(ax+3)^5 = 1215$, therefore:

$$x^3(15a^4)x^4 = 1215x^7$$

$$15a^4 = 1215$$

$$a^4 = 81$$

$$a = \pm\sqrt[4]{81}$$

$$= \pm 3$$

$$\boxed{a = 3 \text{ or } -3}$$

Question 10

Consider the binomial expansion of $\frac{1}{1+x}$.

(a) Write down the first four terms. [4]

(b) Find the values of x such that the complete expansion converges. [2]

(c) Use the terms found in part (a) to estimate $\frac{1}{1.1}$. [2]

(a) By the binomial theorem, for any value of n:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Rewrite $\frac{1}{1+x}$ in the form $(1+x)^n$

$$\frac{1}{1+x} = (1+x)^{\overset{n}{-1}}$$

Substitute values into the formula for $(1+x)^n$

$$(1+x)^{-1} = 1 + (-1)(x) + \frac{(-1)(-2)}{2!}(x)^2 + \frac{(-1)(-2)(-3)}{3!}(x)^3 + \dots$$

$$= 1 - x + \frac{2}{2}x^2 + \frac{(-6)}{6}x^3 + \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\boxed{\text{First four terms: } 1 - x + x^2 - x^3}$$

Consider the binomial expansion of $\frac{1}{1+x}$.

(a) Write down the first four terms. [4]

(b) Find the values of x such that the complete expansion converges. [2]

(c) Use the terms found in part (a) to estimate $\frac{1}{1.1}$. [2]

(b) n is a negative integer, so the series converges when $|x| < 1$

$$|x| < 1$$

$$-1 < x < 1$$

$$\boxed{\text{Converges for } -1 < x < 1}$$

Consider the binomial expansion of $\frac{1}{1+x}$.

(a) Write down the first four terms. [4]

First four terms : $1 - x + x^2 - x^3$

(b) Find the values of x such that the complete expansion converges. [2]

(c) Use the terms found in part (a) to estimate $\frac{1}{1.1}$. [2]

(c) Find the value of x for which $\frac{1}{1+x} = \frac{1}{1.1}$

$$\frac{1}{1+x} = \frac{1}{1.1}$$

$$1 + x = 1.1$$

$$x = 0.1$$

Substitute $x = 0.1$ into the expansion for $(1+x)^{-1}$

$$(1+(0.1))^{-1} = 1 - 0.1 + (0.1)^2 - (0.1)^3 + \ldots$$

$$= 1 - 0.1 + 0.01 - 0.001 + \ldots$$

$$\boxed{\frac{1}{1.1} \approx 0.909}$$

Question 11

Consider the binomial expansion of $\sqrt[3]{4(2+x)}$.

(a) Write down the first three terms. [4]

(b) State the interval of convergence for the complete expansion. [2]

(c) Use the terms found in part (a) to estimate $\sqrt[3]{12}$. Give your answer as a fraction. [2]

(a) Rewrite $\sqrt[3]{4(2+x)}$ in the form $k(1+\frac{x}{a})^n$

$$\sqrt[3]{4(2+x)} = 4^{\frac{1}{3}}(2+x)^{\frac{1}{3}}$$

$$= 4^{\frac{1}{3}}\left[2^{\frac{1}{3}}\left(1+\frac{x}{2}\right)^{\frac{1}{3}}\right]$$

Spot that $4^{\frac{1}{3}} = (2^2)^{\frac{1}{3}} = 2^{\frac{2}{3}}$

$$= 2^{\frac{2}{3}}\left[2^{\frac{1}{3}}\left(1+\frac{x}{2}\right)^{\frac{1}{3}}\right]$$

$$= 2\left(1+\frac{x}{2}\right)^{\frac{1}{3}}$$

Substitute values into the formula for $(1+x)^n$

By the binomial theorem, for any value of n:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$2\left(1+\frac{x}{2}\right)^{\frac{1}{3}} = 2\left[1 + \left(\frac{1}{3}\right)\left(\frac{x}{2}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{2}\right)^2 + \ldots\right]$$

$$= 2\left(1 + \frac{x}{6} + \frac{-\frac{2}{9}}{2}\left(\frac{x^2}{4}\right) + \ldots\right)$$

$$= 2 + \frac{x}{3} - \frac{2}{9}\left(\frac{x^2}{4}\right) + \ldots$$

$$= 2 + \frac{x}{3} - \frac{x^2}{18} + \ldots$$

$$\boxed{\sqrt[3]{4(2+x)} \approx 2 + \frac{x}{3} - \frac{x^2}{18}}$$

Consider the binomial expansion of $\sqrt[3]{4(2+x)}$.

(a) Write down the first three terms. [4]

(b) State the interval of convergence for the complete expansion. [2]

(c) Use the terms found in part (a) to estimate $\sqrt[3]{12}$. Give your answer as a fraction. [2]

(b) $n \geq 0$ and $n \notin \mathbb{N}$, so the series converges when $|x| < 1$

$2\left(1 + \frac{x}{2}\right)^{\frac{1}{3}}$ ← x term

$\left|\frac{x}{2}\right| < 1$

$|x| < 2$

$-2 < x < 2$

$\boxed{\text{Converges for } -2 < x < 2}$

Consider the binomial expansion of $\sqrt[3]{4(2+x)}$.

(a) Write down the first three terms. [4]

$\boxed{\sqrt[3]{4(2+x)} \approx 2 + \frac{x}{3} - \frac{x^2}{18}}$

(b) State the interval of convergence for the complete expansion. [2]

$\boxed{-2 < x < 2}$

(c) Use the terms found in part (a) to estimate $\sqrt[3]{12}$. Give your answer as a fraction. [2]

(c) Find the value of x for which $\sqrt[3]{4(2+x)} = \sqrt[3]{12}$

$\sqrt[3]{4(2+x)} = \sqrt[3]{12}$

$4(2+x) = 12$

$2 + x = 3$

$x = 1$ ← $-2 < x < 2$ so can use the expansion

Substitute $x = 1$ into the expansion for $\sqrt[3]{4(2+x)}$

$\sqrt[3]{4(2+x)} \approx 2 + \frac{x}{3} - \frac{x^2}{18}$

$\sqrt[3]{4(2+1)} \approx 2 + \frac{1}{3} - \frac{1^2}{18}$

$= 2\frac{1}{3} - \frac{1}{18}$

$= 2\frac{6}{18} - \frac{1}{18}$

$\boxed{\sqrt[3]{12} \approx 2\frac{5}{18}}$

Question 12

Consider the binomial expansion of $\frac{1}{\sqrt{4+x}}$.

(a) Write down the first four terms. [4]

(b) State the interval of convergence for the complete expansion. [2]

By the binomial theorem, for any value of n:

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

(a) Rewrite $\frac{1}{\sqrt{4+x}}$ in the form $k(1+\frac{x}{a})^n$

$\frac{1}{\sqrt{4+x}} = \frac{1}{(4+x)^{\frac{1}{2}}} = (4+x)^{-\frac{1}{2}}$

$= 4^{-\frac{1}{2}}(1+\frac{x}{4})^{-\frac{1}{2}}$

$4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

$= \frac{1}{2}(1+\frac{x}{4})^{-\frac{1}{2}}$

Substitute values into the formula for $(1+x)^n$

$\frac{1}{2}(1+\frac{x}{4})^{-\frac{1}{2}} \approx \frac{1}{2}\left[1 + (-\frac{1}{2})(\frac{x}{4}) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(\frac{x}{4})^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(\frac{x}{4})^3\right]$

$= \frac{1}{2}\left[1 + (-\frac{x}{8}) + \frac{(\frac{3}{4})}{2}(\frac{x^2}{16}) + \frac{(-\frac{15}{8})}{6}(\frac{x^3}{64})\right]$

$= \frac{1}{2}\left[1 - \frac{1}{8}x + \frac{3}{128}x^2 - \frac{5}{1024}x^3\right]$

$\frac{1}{2}(1+\frac{x}{4})^{-\frac{1}{2}} = \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 - \frac{5}{2048}x^3 + \ldots$

$$\boxed{\frac{1}{\sqrt{4+x}} \approx \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 - \frac{5}{2048}x^3}$$

Consider the binomial expansion of $\frac{1}{\sqrt{4+x}}$.

(a) Write down the first four terms. [4]

(b) State the interval of convergence for the complete expansion. [2]

(b)

$\left|\frac{x}{4}\right| < 1$

$|x| < 4$

$$\boxed{-4 < x < 4}$$