Question 1

For each of the following, either show that the limit converges and find its value, or else explain why the limit diverges:

(a)

$$\lim_{x\to 4} \frac{1}{x^2 - 9}$$

[2]

(b)

$$\lim_{x\to 3} \frac{1}{x^2 - 9}$$

[2]

(c)

$$\lim_{x\to 3} \frac{x-3}{x^2 - 9}$$

[3]

a) sub in $x = 4$

$\frac{1}{(4)^2 - 9} = \frac{1}{7}$, which is well-defined

$\boxed{\lim_{x\to 4} \frac{1}{x^2-9} = \frac{1}{7}, \text{ so the limit converges}}$

For each of the following, either show that the limit converges and find its value, or else explain why the limit diverges:

(a)

$$\lim_{x\to 4} \frac{1}{x^2 - 9}$$

[2]

(b)

$$\lim_{x\to 3} \frac{1}{x^2 - 9}$$

[2]

(c)

$$\lim_{x\to 3} \frac{x-3}{x^2 - 9}$$

[3]

b) sub in $x = 3$

$\frac{1}{(3)^2 - 9} = \frac{1}{0}$, which is undefined

$\boxed{\lim_{x\to 3} \frac{1}{x^2-9} \text{ diverges to } \pm\infty}$

EXAM PAPERS PRACTICE

For each of the following, either show that the limit converges and find its value, or else explain why the limit diverges:

(a)

$$\lim_{x \to 4} \frac{1}{x^2 - 9}$$

[2]

(b)

$$\lim_{x \to 3} \frac{1}{x^2 - 9}$$

[2]

(c)

$$\lim_{x \to 3} \frac{x - 3}{x^2 - 9}$$

[3]

c) sub in $x = 3$

$\frac{(3) - 3}{(3)^2 - 9} = \frac{0}{0}$, which is undefined

However $\frac{x - 3}{x^2 - 9} = \frac{\cancel{x - 3}}{(\cancel{x - 3})(x + 3)} = \frac{1}{x + 3}$

and $\frac{1}{(3) + 3} = \frac{1}{6}$

$\boxed{\lim_{x \to 3} \frac{x - 3}{x^2 - 9} = \frac{1}{6} \text{ and the limit converges}}$

Note: Due to the indeterminate $\frac{0}{0}$ this could also be done with L'Hôpital's rule.

Question 2

(a) Evaluate the limit

$$\lim_{x \to -\infty} \left(13 - \frac{619}{x^2}\right)$$

justifying your answer by clear mathematical reasoning.

[2]

a) As $x \to -\infty$ $\therefore x^2 \to \infty$ $\therefore \frac{619}{x^2} \to 0$

$\therefore \lim_{x \to -\infty} \left(13 - \frac{619}{x^2}\right) = 13 - \frac{619}{\infty} = 13 - 0$

$\boxed{\lim_{x \to -\infty} \left(13 - \frac{619}{x^2}\right) = 13}$

(b) Show that the limit

$$\lim_{x \to +\infty} \frac{3x^2 - 5x + 7}{x^2}$$

converges, and find its value. Be sure to show clear algebraic working.

[3]

EXAM PAPERS PRACTICE

(a) Evaluate the limit

$$\lim_{x \to -\infty} \left(13 - \frac{619}{x^2}\right)$$

justifying your answer by clear mathematical reasoning.

[2]

(b) Show that the limit

$$\lim_{x \to +\infty} \frac{3x^2 - 5x + 7}{x^2}$$

converges, and find its value. Be sure to show clear algebraic working.

[3]

b) Numerator and denominator diverge to $+\infty$ as $x \to \infty$, which isn't helpful.

$\therefore$ Rearrange into separate fractions.

$$\frac{3x^2 - 5x + 7}{x^2} = 3 - \frac{5}{x} + \frac{7}{x^2}$$

$$\therefore \lim_{x \to \infty} \left(\frac{3x^2 - 5x + 7}{x^2}\right) = \lim_{x \to \infty} \left(3 - \frac{5}{x} + \frac{7}{x^2}\right)$$

last two terms $\to 0$ as $x \to \infty$

$$\boxed{\therefore \lim_{x \to \infty} \left(\frac{3x^2 - 5x + 7}{x^2}\right) = 3 - 0 - 0 = 3}$$

Note: Due to the indeterminate $\frac{+\infty}{+\infty}$ this could also be done with L'Hôpital's rule.

Question 3

A student has attempted to evaluate the limit

$$\lim_{x \to +\infty} (x^3 - x)$$

as follows:

$$\lim_{x \to +\infty} (x^3 - x) = (+\infty)^3 - (+\infty) = (+\infty) - (+\infty) = 0$$

(a) Explain what is wrong with the student's work.

[2]

(b) Determine the correct evaluation of the limit, justifying your answer by clear mathematical reasoning.

[2]

(c) Use technology to help you sketch the graph of $y = x^3 - x$, and show that the graph confirms your answer to part (b).

[2]

a) As $x \to \infty$, it is true that both x^3 and x tend to $+\infty$. However, '$+\infty$' isn't a number, but rather an idea or an imagined process that is never complete. So you can't subtract $+\infty$ from $+\infty$ to get zero.

A student has attempted to evaluate the limit

$$\lim_{x \to +\infty} (x^3 - x)$$

as follows:

$$\lim_{x \to +\infty} (x^3 - x) = (+\infty)^3 - (+\infty) = (+\infty) - (+\infty) = 0$$

(a) Explain what is wrong with the student's work.

[2]

(b) Determine the correct evaluation of the limit, justifying your answer by clear mathematical reasoning.

[2]

(c) Use technology to help you sketch the graph of $y = x^3 - x$, and show that the graph confirms your answer to part (b).

[2]

b) Best way to do this is to factorise $x^3 - x = x(x^2 - 1)$, once $x > 1$ then both terms are positive, which gives a positive result.

$$\boxed{\lim_{x \to \infty} (x^3 - x) = \lim_{x \to \infty} (x(x^2 - 1)) = +\infty}$$

A student has attempted to evaluate the limit

$$\lim_{x \to +\infty} (x^3 - x)$$

as follows:

$$\lim_{x \to +\infty} (x^3 - x) = (+\infty)^3 - (+\infty) = (+\infty) - (+\infty) = 0$$

(a) Explain what is wrong with the student's work.

[2]

(b) Determine the correct evaluation of the limit, justifying your answer by clear mathematical reasoning.

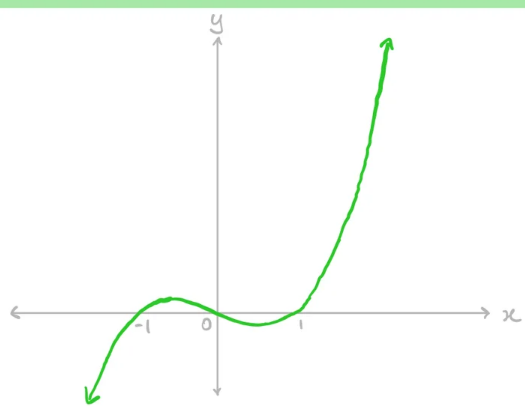
$$\boxed{\lim_{x \to \infty} (x^3 - x) = \lim_{x \to \infty} (x(x^2 - 1)) = +\infty}$$

[2]

(c) Use technology to help you sketch the graph of $y = x^3 - x$, and show that the graph confirms your answer to part (b).

[2]

c) Sketch enough of the graph to show it tends to ∞ as x increases.



we can see the graph tends to ∞ as x increases.

EXAM PAPERS PRACTICE

Question 4

Consider the function f defined by

$$f(x) = \frac{1}{x^2}$$

(a) Evaluate the limits

(i) $\lim_{x \to 0^-} f(x)$

(ii) $\lim_{x \to 0^+} f(x)$

[3]

(b) Evaluate the limits

(i) $\lim_{x \to -\infty} f(x)$

(ii) $\lim_{x \to +\infty} f(x)$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = f(x)$.

[2]

(d) Use technology to help you sketch the graph of $y = f(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

a) i) For $x < 0$, $x^2 > 0$ $\therefore \frac{1}{x^2} > 0$

$\therefore$ as $x \to 0$ from the negative side,

$\frac{1}{x^2} \to +\frac{1}{0}$

$$\boxed{\lim_{x \to 0^-} f(x) = +\infty}$$

ii) For $x > 0$, $x^2 > 0$ $\therefore \frac{1}{x^2} > 0$

$\therefore$ as $x \to 0$ from the positive side,

$\frac{1}{x^2} \to +\frac{1}{0}$

$$\boxed{\lim_{x \to 0^+} f(x) = +\infty}$$

Because the + and - limits agree, this means that $\lim_{x \to 0} f(x)$ is relatively well-defined as $+\infty$.

Consider the function f defined by

$$f(x) = \frac{1}{x^2}$$

(a) Evaluate the limits

(i) $\lim_{x \to 0^-} f(x)$

(ii) $\lim_{x \to 0^+} f(x)$

[3]

(b) Evaluate the limits

(i) $\lim_{x \to -\infty} f(x)$

(ii) $\lim_{x \to +\infty} f(x)$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = f(x)$.

[2]

(d) Use technology to help you sketch the graph of $y = f(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

b) i) For $x < 0$, $x^2 > 0$ $\therefore \frac{1}{x^2} > 0$

$\therefore$ as $x \to -\infty$, $\frac{1}{x^2} \to +\frac{1}{\infty}$

$$\boxed{\lim_{x \to -\infty} f(x) = 0}$$

ii) For $x > 0$, $x^2 > 0$ $\therefore \frac{1}{x^2} > 0$

$\therefore$ as $x \to +\infty$, $\frac{1}{x^2} \to +\frac{1}{\infty}$

$$\boxed{\lim_{x \to +\infty} f(x) = 0}$$

Consider the function f defined by

$$f(x) = \frac{1}{x^2}$$

(a) Evaluate the limits

(i) $\lim_{x\to 0^-} f(x)$

$\boxed{\lim_{x\to 0^-} f(x) = +\infty}$

(ii) $\lim_{x\to 0^+} f(x)$

$\boxed{\lim_{x\to 0^+} f(x) = +\infty}$

[3]

(b) Evaluate the limits

(i) $\lim_{x\to -\infty} f(x)$

$\boxed{\lim_{x\to -\infty} f(x) = 0}$

(ii) $\lim_{x\to +\infty} f(x)$

$\boxed{\lim_{x\to +\infty} f(x) = 0}$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = f(x)$.

[2]

(d) Use technology to help you sketch the graph of $y = f(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

c) The line $x = c$ is a vertical asymptote of the graph $y = f(x)$ if either $\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$.

The line $y = k$ is a horizontal asymptote of $f(x)$ if either $\lim_{x\to -\infty} f(x) = k$ or $\lim_{x\to +\infty} f(x) = k$.

$\therefore$ $f(x)$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.

Consider the function f defined by

$$f(x) = \frac{1}{x^2}$$

(a) Evaluate the limits

(i) $\lim_{x\to 0^-} f(x)$

(ii) $\lim_{x\to 0^+} f(x)$

[3]

(b) Evaluate the limits

(i) $\lim_{x\to -\infty} f(x)$

(ii) $\lim_{x\to +\infty} f(x)$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = f(x)$.

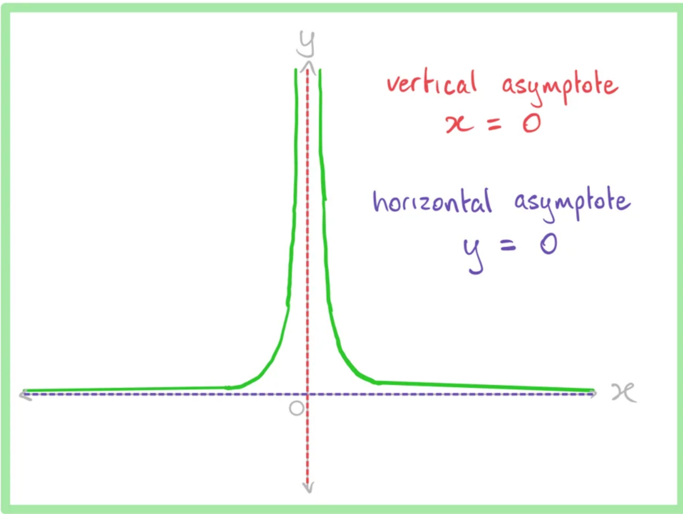
vertical asymptote $x = 0$, horizontal asymptote $y = 0$.

[2]

(d) Use technology to help you sketch the graph of $y = f(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

d) Sketch the graph showing its limiting behaviour at $x = 0$ and as $x \to \pm\infty$.



Question 5

Consider the function g defined by

$$g(x) = \frac{1}{x-5}$$

(a) Evaluate the limits

(i) $\lim_{x \to 5^-} g(x)$

(ii) $\lim_{x \to 5^+} g(x)$

[3]

(b) Evaluate the limits

(i) $\lim_{x \to -\infty} g(x)$

(ii) $\lim_{x \to +\infty} g(x)$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = g(x)$.

[2]

(d) Use technology to help you sketch the graph of $y = g(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

a) i) For $x < 5$, $x - 5 < 0$ $\therefore \frac{1}{x-5} < 0$

$\therefore$ as $x \to 5$ from the negative side,

$\frac{1}{x-5} \longrightarrow -\frac{1}{0}$

$\boxed{\lim_{x \to 5^-} g(x) = -\infty}$

ii) For $x > 5$, $x - 5 > 0$ $\therefore \frac{1}{x-5} > 0$

$\therefore$ as $x \to 5$ from the positive side,

$\frac{1}{x-5} \longrightarrow +\frac{1}{0}$

$\boxed{\lim_{x \to 5^+} g(x) = +\infty}$

Because the + and − limits do not agree, this means that $\lim_{x \to 5} g(x)$ is undefined.

Consider the function g defined by

$$g(x) = \frac{1}{x-5}$$

(a) Evaluate the limits

(i) $\lim_{x \to 5^-} g(x)$

(ii) $\lim_{x \to 5^+} g(x)$

[3]

(b) Evaluate the limits

(i) $\lim_{x \to -\infty} g(x)$

(ii) $\lim_{x \to +\infty} g(x)$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = g(x)$.

[2]

(d) Use technology to help you sketch the graph of $y = g(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

b) i) For $x < 5$, $x - 5 < 0$ $\therefore \frac{1}{x-5} < 0$

$\therefore$ as $x \to -\infty$, $\frac{1}{x-5} \longrightarrow -\frac{1}{\infty}$

$\boxed{\lim_{x \to -\infty} g(x) = 0}$

ii) For $x > 5$, $x - 5 > 0$ $\therefore \frac{1}{x-5} > 0$

$\therefore$ as $x \to +\infty$, $\frac{1}{x-5} \longrightarrow +\frac{1}{\infty}$

$\boxed{\lim_{x \to +\infty} g(x) = 0}$

The first of these approaches zero from below, and the second approaches zero from above.

Consider the function g defined by

$$g(x) = \frac{1}{x-5}$$

(a) Evaluate the limits

(i) $\lim_{x\to5^-} g(x)$

$\boxed{\lim_{x\to5^-} g(x) = -\infty}$

(ii) $\lim_{x\to5^+} g(x)$

$\boxed{\lim_{x\to5^+} g(x) = +\infty}$

[3]

(b) Evaluate the limits

(i) $\lim_{x\to-\infty} g(x)$

$\boxed{\lim_{x\to-\infty} g(x) = 0}$

(ii) $\lim_{x\to+\infty} g(x)$

$\boxed{\lim_{x\to+\infty} g(x) = 0}$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = g(x)$.

[2]

(d) Use technology to help you sketch the graph of $y = g(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

c) The line $x = c$ is a vertical asymptote of the graph $y = g(x)$ if either $\lim_{x\to c^+} g(x) = \pm\infty$ or $\lim_{x\to c^-} g(x) = \pm\infty$.

The line $y = k$ is a horizontal asymptote of $g(x)$ if either $\lim_{x\to-\infty} g(x) = k$ or $\lim_{x\to+\infty} g(x) = k$.

$\therefore$ $g(x)$ has a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 0$.

Consider the function g defined by

$$g(x) = \frac{1}{x-5}$$

(a) Evaluate the limits

(i) $\lim_{x\to5^-} g(x)$

(ii) $\lim_{x\to5^+} g(x)$

[3]

(b) Evaluate the limits

(i) $\lim_{x\to-\infty} g(x)$

(ii) $\lim_{x\to+\infty} g(x)$

[3]

(c) Use your results from parts (a) and (b) to write down the equations of any asymptotes on the graph of $y = g(x)$.

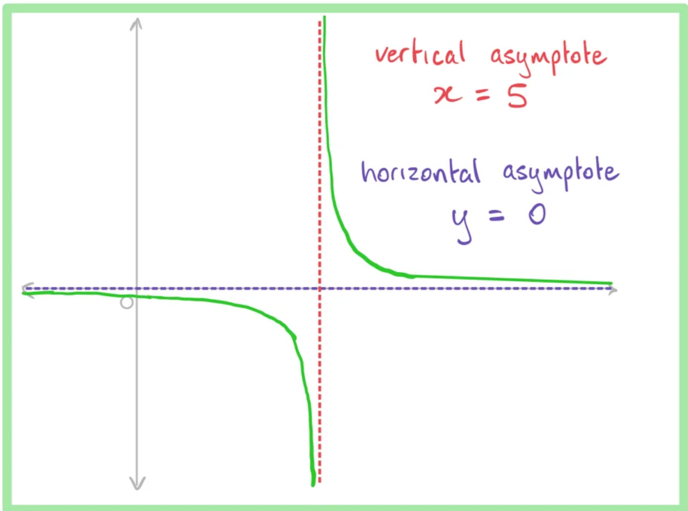
$\boxed{\text{vertical asymptote } x = 5, \text{ horizontal asymptote } y = 0.}$

[2]

(d) Use technology to help you sketch the graph of $y = g(x)$, and show that this confirms your results from parts (a), (b) and (c).

[2]

d) Sketch the graph showing its limiting behaviour at $x = 0$ and as $x \to \pm\infty$.



Question 6

(a) The function f is a piecewise function defined by

$$f(x) = \begin{cases} x^2, & x \leq 2 \\ x+3, & x > 2 \end{cases}$$

Explain why f is not continuous at $x = 2$.

[3]

(b) A function g is defined for all $x \in \mathbb{R}$, and it is differentiable at all points $x \in \mathbb{R}$.

Explain why g is continuous at $x = 7$.

[2]

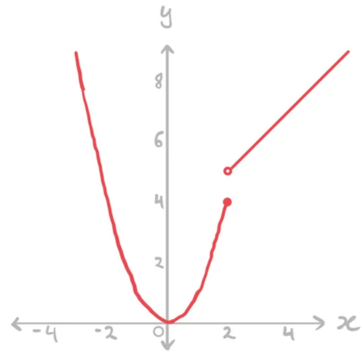
a) Method 1: Formal use of limits

$f(2) = (2)^2 = 4$, however

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x+3) = 5 \neq 4$

$\boxed{\therefore f \text{ is not continuous at } x = 2.}$

Method 2: Sketch a graph.



if f was continuous it would not have the gap at $x = 2$.

$\boxed{\therefore f \text{ is not continuous at } x = 2.}$

(a) The function f is a piecewise function defined by

$$f(x) = \begin{cases} x^2, & x \leq 2 \\ x+3, & x > 2 \end{cases}$$

Explain why f is not continuous at $x = 2$.

[3]

(b) A function g is defined for all $x \in \mathbb{R}$, and it is differentiable at all points $x \in \mathbb{R}$.

Explain why g is continuous at $x = 7$.

[2]

b) If a function is differentiable at a point in its domain then it is continuous at that point. $x = 7$ is in the domain of g, and g is differentiable there.
$\therefore g$ is continuous at $x = 7$.

Note: Differentiability is a stronger condition than continuity. If a point is differentiable, then it is continuous. However if a point is continuous it is not necessarily differentiable.