Question 1

Find the following indefinite integrals:

(a)

$$\int 4\sec^2 2x \, dx$$

[2]

(b)

$$\int \sec\frac{x}{3}\tan\frac{x}{3} \, dx$$

[2]

(c)

$$\int \frac{1}{\sin^2\left(x+\frac{\pi}{4}\right)} \, dx$$

[3]

(a) Integration by substitution

Let $u = 2x \Rightarrow \frac{du}{dx} = 2$

$du = 2\,dx$

$\Rightarrow \frac{1}{2}du = dx$

Rewrite original integral using the substitutions

$$\int 4\sec^2(u)\left(\frac{1}{2}\right)du = \int 2\sec^2(u)\,du$$

$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$ ← Formula booklet

$\Rightarrow \int \sec^2 x = \tan x + C$

$$\int 2\sec^2(u)\,du = 2\tan u + C$$

$$\boxed{\int 4\sec^2 2x \, dx = 2\tan(2x) + C}$$

Find the following indefinite integrals:

(a)

$$\int 4\sec^2 2x \, dx$$

[2]

(b)

$$\int \sec\frac{x}{3}\tan\frac{x}{3} \, dx$$

[2]

(c)

$$\int \frac{1}{\sin^2\left(x+\frac{\pi}{4}\right)} \, dx$$

[3]

(a) Integration by substitution

Let $u = \frac{1}{3}x \Rightarrow \frac{du}{dx} = \frac{1}{3}$

$du = \frac{1}{3}dx$

$\Rightarrow dx = 3\,du$

Rewrite original integral using the substitutions

$$\int \sec(u)\tan(u) \times 3\,du$$

$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$ ← Formula booklet

$\Rightarrow \int \sec x \tan x = \sec x + C$

$$\int \sec(u)\tan(u) \times 3\,du = 3\sec(u) + C$$

$$\boxed{\int \sec\frac{x}{3}\tan\frac{x}{3}\,dx = 3\sec\left(\frac{1}{3}x\right) + C}$$

EXAM PAPERS PRACTICE

Find the following indefinite integrals:

(a)

$$\int 4\sec^2 2x \ dx$$

[2]

(b)

$$\int \sec\frac{x}{3}\tan\frac{x}{3} \ dx$$

[2]

(c)

$$\int \frac{1}{\sin^2\left(x+\frac{\pi}{4}\right)} \ dx$$

[3]

(c) Rewrite the Integral as a reciprocal function

$$\int \operatorname{cosec}^2\left(x+\frac{\pi}{4}\right) dx$$

Integration by substitution

Let $u = x + \frac{\pi}{4} \Rightarrow \frac{du}{dx} = 1$

$\Rightarrow du = dx$

Rewrite the integral using the substitutions

$$\int \operatorname{cosec}^2(u)\, du$$

$f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$ ← Formula booklet

$\Rightarrow \int \operatorname{cosec}^2 x = -\cot x + C$

$$\int \operatorname{cosec}^2(u)\, du = -\cot(u) + C$$

$$\boxed{\int \frac{1}{\sin^2\left(x+\frac{\pi}{4}\right)} dx = -\cot\left(x+\frac{\pi}{4}\right) + C}$$

Question 2

Find the following indefinite integrals:

(a)

$$\int (\ln 3)3^x \ dx$$

[2]

(b)

$$\int \frac{12}{9+x^2} \ dx$$

[2]

(c)

$$\int \frac{3}{5\sqrt{16-x^2}} \ dx$$

[2]

(d) Using a sketch, briefly describe the family of graphs corresponding to all the possible specific solutions to the integral in part (a).

[3]

(a) $\int a^x \, dx = \frac{1}{\ln a} a^x + C$ ← Formula booklet

$$(\ln 3)\int 3^x \, dx = (\ln 3)\frac{1}{\ln 3} 3^x + C$$

$$\boxed{\int (\ln 3)3^x \, dx = 3^x + C}$$

Find the following indefinite integrals:

(a)

$$\int (\ln 3)3^x \, dx$$

[2]

(b)

$$\int \frac{12}{9+x^2} \, dx$$

[2]

(c)

$$\int \frac{3}{5\sqrt{16-x^2}} \, dx$$

[2]

(d) Using a sketch, briefly describe the family of graphs corresponding to all the possible specific solutions to the integral in part (a).

[3]

(b) Factorise the denominator and split up the fraction

$$\int \frac{12}{9\left(1+\frac{x^2}{9}\right)} dx = \frac{12}{9}\int \frac{1}{1+\frac{x^2}{9}} dx$$

$$= \frac{4}{3}\int \frac{1}{1+\left(\frac{x}{3}\right)^2} dx$$

Integration by substitution

Let $u = \frac{x}{3} \Rightarrow \frac{du}{dx} = \frac{1}{3}$

$$du = \frac{1}{3} dx$$

$$\Rightarrow dx = 3\,du$$

Rewrite the original integral using the substitutions

$$\frac{4}{3}\int \frac{1}{1+(u)^2} \times 3\,du = 4\int \frac{1}{1+(u)^2} du$$

Find the following indefinite integrals:

(a)

$$\int (\ln 3)3^x \, dx$$

[2]

(b)

$$\int \frac{12}{9+x^2} \, dx$$

[2]

(c)

$$\int \frac{3}{5\sqrt{16-x^2}} \, dx$$

[2]

(d) Using a sketch, briefly describe the family of graphs corresponding to all the possible specific solutions to the integral in part (a).

[3]

(c) Split the fraction

$$\int \frac{3}{5} \times \frac{1}{\sqrt{16-x^2}} dx = \frac{3}{5}\int \frac{1}{\sqrt{4^2-x^2}} dx$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$$ ← Formula booklet

$$\boxed{\int \frac{3}{5\sqrt{16-x^2}} dx = \frac{3}{5}\arcsin\left(\frac{x}{4}\right) + C}$$

Find the following indefinite integrals:

(a)

$$\int (\ln 3)3^x \, dx$$

[2]

(b)

$$\int \frac{12}{9+x^2} \, dx$$

[2]

(c)

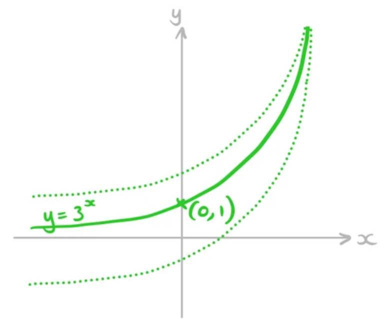
$$\int \frac{3}{5\sqrt{16-x^2}} \, dx$$

[2]

(d) Using a sketch, briefly describe the family of graphs corresponding to all the possible specific solutions to the integral in part (a).

[3]

(d)



The graphs described by $y = 3^x + c$ are a family of curves that are a vertical translation of the graph $y = 3^x$

Question 3

(a) Show that

$$\frac{5}{x^2+9x+14} = \frac{A}{x+2} + \frac{B}{x+7}$$

where A and B are constants to be found.

[2]

(b) Hence find the indefinite integral

$$\int \frac{5}{x^2+9x+14} \, dx$$

using laws of logarithms to simplify your answer as far as possible.

[4]

(c) Show that

$$\int_{-1}^{5} \frac{5}{x^2+9x+14} \, dx = \ln 7 - \ln 2$$

[4]

(a) Add together the fractions on the RHS

$$\frac{5}{x^2+9x+14} = \frac{A(x+7) + B(x+2)}{(x+7)(x+2)}$$

$$\frac{5}{x^2+9x+14} = \frac{Ax + 7A + Bx + 2B}{x^2+9x+14}$$

$$\Rightarrow 5 = (A+B)x + (7A + 2B)$$

Compare the terms on both sides

$(A+B)x = 0$ and $(7A+2B) = 5$

$\Rightarrow A = -B$ $\qquad 7(-B) + 2B = 5$

$-5B = 5$

$B = -1$

$\Rightarrow A = 1$

$$\boxed{\frac{5}{x^2+9x+14} = \frac{1}{x+2} - \frac{1}{x+7}}$$

(a) Show that

$$\frac{5}{x^2+9x+14}=\frac{A}{x+2}+\frac{B}{x+7}$$

where A and B are constants to be found.

$$\boxed{\frac{5}{x^2+9x+14}=\frac{1}{x+2}-\frac{1}{x+7}}$$

[2]

(b) Hence find the indefinite integral

$$\int\frac{5}{x^2+9x+14}\,dx$$

using laws of logarithms to simplify your answer as far as possible.

[4]

(c) Show that

$$\int_{-1}^{5}\frac{5}{x^2+9x+14}\,dx=\ln 7-\ln 2$$

[4]

(b) $\displaystyle\int\frac{1}{x+2}-\frac{1}{x+7}\,dx$

$= \ln|x+2|-\ln|x+7|+C$

Remember to include the modulus signs so that all values of x (except for $x=-2$ or $x=-7$) are defined

$$\boxed{\ln\left|\frac{x+2}{x+7}\right|+C}$$

(a) Show that

$$\frac{5}{x^2+9x+14}=\frac{A}{x+2}+\frac{B}{x+7}$$

where A and B are constants to be found.

[2]

(b) Hence find the indefinite integral

$$\int\frac{5}{x^2+9x+14}\,dx$$

using laws of logarithms to simplify your answer as far as possible.

$$\boxed{\ln\left|\frac{x+2}{x+7}\right|+C}$$

[4]

(c) Show that

$$\int_{-1}^{5}\frac{5}{x^2+9x+14}\,dx=\ln 7-\ln 2$$

[4]

(c) $\displaystyle\int_{-1}^{5}\frac{1}{x+2}-\frac{1}{x+7}\,dx = \left[\ln\frac{|x+2|}{|x+7|}\right]_{-1}^{5}$

$= \ln\frac{|5+2|}{|5+7|}-\ln\frac{|-1+2|}{|-1+7|}$

$= \ln\left(\frac{7}{12}\right)-\ln\left(\frac{1}{6}\right)$

$= \ln\left(\frac{7}{12}\times\frac{6}{1}\right)$

$= \ln\left(\frac{7}{2}\right)$

$$\boxed{\int_{-1}^{5}\frac{5}{x^2+9x+14}\,dx=\ln 7-\ln 2}$$

Question 4

(a) Use the substitution $u = x - 3$ to find the following indefinite integral:

$$\int x\sqrt{x-3}\,dx$$

[4]

(b) Hence find the value of the definite integral

$$\int_4^7 x\sqrt{x-3}\,dx$$

(i) by evaluating the definite integral entirely in terms of x

(ii) by converting the integral limits to appropriate values of u and evaluating the definite integral entirely in terms of u.

Verify that the two methods give the same result for the value of the integral.

[4]

(a) Let $u = x - 3 \Rightarrow x = u + 3$

$\frac{du}{dx} = 1 \Rightarrow dx = du$

Rewrite the integral with the substitutions

$$\int (u+3)\sqrt{u}\,du = \int (u+3)u^{1/2}\,du$$

$$= \int u^{3/2} + 3u^{1/2}\,du$$

Integrate

$$= \frac{2}{5}u^{5/2} + 3 \times \frac{2}{3}u^{3/2} + c$$

$$= \frac{2}{5}u^{5/2} + 2u^{3/2} + c$$

Replace $u = x - 3$

$$\boxed{\int \sqrt{x-3}\,dx = \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + c}$$

(a) Use the substitution $u = x - 3$ to find the following indefinite integral:

$$\int x\sqrt{x-3}\,dx$$

$\boxed{\frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + c}$ $\boxed{\frac{2}{5}u^{5/2} + 2u^{3/2} + c}$

[4]

(b) Hence find the value of the definite integral

$$\int_4^7 x\sqrt{x-3}\,dx$$

(i) by evaluating the definite integral entirely in terms of x

(ii) by converting the integral limits to appropriate values of u and evaluating the definite integral entirely in terms of u.

Verify that the two methods give the same result for the value of the integral.

[4]

(b) (i) $$\int_{x=4}^{x=7} x\sqrt{x-3}\,dx = \left[\frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2}\right]_{x=4}^{x=7}$$

$$= \left[\frac{2}{5}(7-3)^{5/2} + 2(7-3)^{3/2}\right] - \left[\frac{2}{5}(4-3)^{5/2} + 2(4-3)^{3/2}\right]$$

$$= \left(\frac{2}{5}\sqrt{4}^5 + 2\sqrt{4}^3\right) - \left(\frac{2}{5} + 2\right) = \boxed{\frac{132}{5}}$$

(ii) Adjust the integral limits

Given that $u = x - 3$, $x = 7 \Rightarrow u = 4$
$x = 4 \Rightarrow u = 1$

Write the integral in terms of u

$$\int_{x=4}^{x=7} x\sqrt{x-3}\,dx = \int_{u=1}^{u=4} (u+3)u^{1/2}\,du = \left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right]_{u=1}^{u=4}$$

$$= \left[\frac{2}{5}(4)^{5/2} + 2(4)^{3/2}\right] - \left[\frac{2}{5}(1)^{5/2} + 2(1)^{3/2}\right]$$

$$= \left(\frac{2}{5}\sqrt{4}^5 + 2\sqrt{4}^3\right) - \left(\frac{2}{5} + 2\right) = \boxed{\frac{132}{5}}$$

It is often easier to keep everything in terms of u when you are calculating the definite integral

Both methods give the same result for the definite integral

Question 5

(a) Show that $x^2 - 10x + 29$ may be written in the form $p + (x - q)^2$, where p and q are constants to be determined.

[2]

(b) Using your results from part (a) along with the substitution $u = x - q$, show that

$$\int \frac{1}{x^2 - 10x + 29}\,\mathrm{d}x = \frac{1}{2}\arctan\left(\frac{x-5}{2}\right) + c$$

[5]

(c) Find the exact value of the definite integral

$$\int_5^7 \frac{1}{x^2 - 10x + 29}\,\mathrm{d}x$$

[3]

(a) $x^2 - 10x + 29 = (x-5)^2 - 25 + 29$

$$\boxed{x^2 - 10x + 29 = 4 + (x-5)^2}$$

(a) Show that $x^2 - 10x + 29$ may be written in the form $p + (x - q)^2$, where p and q are constants to be determined.

$$\boxed{4 + (x-5)^2}$$

[2]

(b) Using your results from part (a) along with the substitution $u = x - q$, show that

$$\int \frac{1}{x^2 - 10x + 29}\,\mathrm{d}x = \frac{1}{2}\arctan\left(\frac{x-5}{2}\right) + c$$

[5]

(c) Find the exact value of the definite integral

$$\int_5^7 \frac{1}{x^2 - 10x + 29}\,\mathrm{d}x$$

[3]

(b) Rewrite the integral using part (a)

$$\int \frac{1}{x^2 - 10x + 29}\,dx = \int \frac{1}{4 + (x-5)^2}\,dx$$

Let $u = x - 5 \Rightarrow \frac{du}{dx} = 1$

$dx = du$

Rewrite the integral with the substitutions

$$\int \frac{1}{4 + u^2}\,du = \int \frac{1}{2^2 + u^2}\,du$$

$$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$ ← Formula booklet

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$

$$\boxed{\int \frac{1}{x^2 - 10x + 29}\,dx = \frac{1}{2}\arctan\left(\frac{x-5}{2}\right) + C}$$

(a) Show that $x^2 - 10x + 29$ may be written in the form $p + (x-q)^2$, where p and q are constants to be determined.

[2]

(b) Using your results from part (a) along with the substitution $u = x - q$, show that

$$\int \frac{1}{x^2 - 10x + 29}\,dx = \frac{1}{2}\arctan\left(\frac{x-5}{2}\right) + c$$

[5]

(c) Find the exact value of the definite integral

$$\int_5^7 \frac{1}{x^2 - 10x + 29}\,dx$$

[3]

(c) $\displaystyle\int_5^7 \frac{1}{x^2 - 10x + 29}\,dx = \left[\frac{1}{2}\arctan\left(\frac{x-5}{2}\right)\right]_5^7$

$= \left[\frac{1}{2}\arctan\left(\frac{7-5}{2}\right)\right] - \left[\frac{1}{2}\arctan\left(\frac{5-5}{2}\right)\right]$

$= \frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0)$

Exact trig values:
$\tan\left(\frac{\pi}{4}\right) = 1$
$\tan(0) = 0$

$= \frac{1}{2}\left(\frac{\pi}{4}\right) - 0$

$$\boxed{\int_5^7 \frac{1}{x^2 - 10x + 29}\,dx = \frac{\pi}{8}}$$

Question 6

Use the substitution $u = 1 + \sin^3 x$ to show that

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \sqrt{1 + \sin^3 x}\,dx = \frac{2}{9}(2\sqrt{2} - 1)$$

[6]

Use the chain rule to differentiate

Let $u = 1 + \sin^3 x \Rightarrow \frac{du}{dx} = 3\sin^2 x \cos x$

$dx = \frac{1}{3\sin^2 x \cos x}\,du$

Rewrite the integral with the substitutions

$\displaystyle\int_{x=0}^{x=\frac{\pi}{2}} \cancel{\sin^2 x \cos x} \sqrt{u}\,\frac{1}{3\cancel{\sin^2 x \cos x}}\,du = \int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{3} u^{1/2}\,du$

Convert the integral limits

$u = 1 + \sin^3 x$

Exact trig values: $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin(0) = 0$

when $x = \frac{\pi}{2} \Rightarrow u = 2$

$x = 0 \Rightarrow u = 1$

Evaluate the definite integral using the new limits

$\displaystyle\int_{u=1}^{u=2} \frac{1}{3} u^{1/2}\,du = \left[\frac{2}{9} u^{3/2}\right]_{u=1}^{u=2}$

$= \left[\frac{2}{9}(2)^{3/2}\right] - \left[\frac{2}{9}(1)^{3/2}\right]$

$= \frac{2}{9} \times 2\sqrt{2} - \frac{2}{9}$

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \sqrt{1 + \sin^3 x}\,dx = \frac{2}{9}\left(2\sqrt{2} - 1\right)}$$

Note: If the question didn't specify to use substitution and you notice that $\sin^2 x \cos x$ is a multiple of the derivative of $1 + \sin^3 x$, you could use the reverse chain rule to integrate the function

Question 7

(a) Use integration by parts to find the indefinite integral

$$\int xe^{2x}\,dx$$

[4]

(b) Hence find the exact value of the definite integral

$$\int_0^3 xe^{2x}\,dx$$

[3]

(c) Use technology to evaluate the integral in part (b), and compare this to the exact value you found.

[1]

(a) $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ ← Formula booklet

Let $u = x$ $\quad \frac{dv}{dx} = e^{2x}$

$\Rightarrow \frac{du}{dx} = 1$ $\quad \Rightarrow v = \frac{1}{2}e^{2x}$

Rewrite the integral with the substitutions

$$\int xe^{2x}\,dx = x \times \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \times 1\,dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$\boxed{\int xe^{2x}\,dx = \frac{1}{4}e^{2x}(2x-1) + C}$$

(a) Use integration by parts to find the indefinite integral

$$\int xe^{2x}\,dx \quad \boxed{\frac{1}{4}e^{2x}(2x-1) + C}$$

[4]

(b) Hence find the exact value of the definite integral

$$\int_0^3 xe^{2x}\,dx$$

[3]

(c) Use technology to evaluate the integral in part (b), and compare this to the exact value you found.

[1]

(b) $\int_0^3 xe^{2x}\,dx = \left[\frac{1}{4}e^{2x}(2x-1)\right]_0^3$

$$= \left[\frac{1}{4}e^{2(3)}(2(3)-1)\right] - \left[\frac{1}{4}e^{2(0)}(2(0)-1)\right]$$

$$= \frac{5}{4}e^6 + \frac{1}{4}$$

$$\boxed{\int_0^3 xe^{2x}\,dx = \frac{1}{4}(5e^6+1)}$$

(a) Use integration by parts to find the indefinite integral

$$\int xe^{2x}\,dx$$

[4]

(b) Hence find the exact value of the definite integral

$$\int_0^3 xe^{2x}\,dx \qquad \boxed{\frac{1}{4}\left(5e^6+1\right)}$$

[3]

(c) Use technology to evaluate the integral in part (b), and compare this to the exact value you found.

[1]

(c) Using the GDC

$$\int_0^3 xe^{2x}\,dx = 504.53599186\ldots$$

The answers to parts (b) and (c) match, $\frac{1}{4}(5e^6+1) = 504.53599\ldots$ but there will always be an approximation with a decimal value rather than an exact value

Question 8

(a) Use integration by parts twice to show that

$$\int 32x^2e^{4x}\,dx = e^{4x}(px^2+qx+r)+c$$

where p, q and r are constants to be found, and where c is a constant of integration.

[6]

Let f be a function defined for all $x \in \mathbb{R}$. Consider the graph of $y = f(x)$.

(b) Given that

$$\frac{dy}{dx} = 32x^2e^{4x}$$

and that the graph passes through the point $\left(\frac{1}{4}, \frac{e+7}{2}\right)$, find an expression for $f(x)$.

[3]

(a) $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ ← Formula booklet

Let $u = 32x^2$ $\quad \frac{dv}{dx} = e^{4x}$

$\Rightarrow \frac{du}{dx} = 64x \quad \Rightarrow \quad v = \frac{1}{4}e^{4x}$

Rewrite the integral with the substitutions

$$\int 32x^2e^{4x}\,dx = (32x^2)\left(\frac{1}{4}e^{4x}\right) - \int (64x)\left(\frac{1}{4}e^{4x}\right)dx$$

$$= 8x^2e^{4x} - \int 16xe^{4x}\,dx$$

Perform integration by parts again

Let $u = 16x$ $\quad \frac{dv}{dx} = e^{4x}$

$\Rightarrow \frac{du}{dx} = 16 \quad \Rightarrow \quad v = \frac{1}{4}e^{4x}$

$$8x^2e^{4x} - \int 16xe^{4x}\,dx$$

$$= 8x^2e^{4x} - \left((16x)\left(\frac{1}{4}e^{4x}\right) - \int (16)\left(\frac{1}{4}e^{4x}\right)dx\right)$$

$$= 8x^2e^{4x} - \left(4xe^{4x} - e^{4x} + c\right)$$

$$= 8x^2e^{4x} - 4xe^{4x} + e^{4x} + c$$

$$\boxed{\int 32x^2e^{4x}\,dx = e^{4x}(8x^2-4x+1) + c}$$

(a) Use integration by parts twice to show that

$$\int 32x^2 e^{4x}\,dx = e^{4x}(px^2 + qx + r) + c$$

where p, q and r are constants to be found, and where c is a constant of integration.

$\boxed{e^{4x}\left(8x^2 - 4x + 1\right) + c}$

[6]

Let f be a function defined for all $x \in \mathbb{R}$. Consider the graph of $y = f(x)$.

(b) Given that

$$\frac{dy}{dx} = 32x^2 e^{4x}$$

and that the graph passes through the point $\left(\frac{1}{4}, \frac{e+7}{2}\right)$, find an expression for $f(x)$.

[3]

(b) $f(x) = \int 32x^2 e^{4x}\,dx$

$\Rightarrow f(x) = e^{4x}\left(8x^2 - 4x + 1\right) + c$

Substitute the given coordinates into $f(x)$ to find c

$$f\left(\frac{1}{4}\right) = \frac{e+7}{2} = e^{4\left(\frac{1}{4}\right)}\left(8\left(\frac{1}{4}\right)^2 - 4\left(\frac{1}{4}\right) + 1\right) + c$$

$$\frac{e+7}{2} = e\left(\frac{1}{2} - 1 + 1\right) + c$$

$$\frac{1}{2}e + \frac{7}{2} = \frac{1}{2}e + c$$

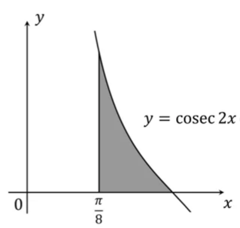
$$c = \frac{7}{2}$$

$$\boxed{f(x) = e^{4x}\left(8x^2 - 4x + 1\right) + \frac{7}{2}}$$

Question 9

Let f be the function defined by $f(x) = \operatorname{cosec} 2x \cot 2x,\ 0 < x < \frac{\pi}{2}$.

The diagram below shows a part of the graph of the curve $y = f(x)$. The shaded region is the region bounded by the curve, the positive x-axis and the line $x = \frac{\pi}{8}$.



Find the exact area of the shaded region.

[6]

Find the area by finding the integral between the limits $\frac{\pi}{8}$ and $f(x) = 0$

$f(x) = 0 = \underbrace{\operatorname{cosec} 2x}\,\underbrace{\cot 2x}$, $\quad x = \frac{\pi}{4}$

$\operatorname{cosec} 2x = \frac{1}{\sin 2x}$

$\Rightarrow$ It will never be equal to 0

$\cot 2x = \frac{\cos 2x}{\sin 2x} = 0$

$\Rightarrow \cos 2x = 0$

$\Rightarrow \quad 2x = \frac{\pi}{2} \quad \Rightarrow \quad x = \frac{\pi}{4}$

Integral limits: $\frac{\pi}{8}$, $\frac{\pi}{4}$

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \operatorname{cosec} 2x \cot 2x \, dx$$

Integration by substitution

Let $\quad u = 2x$

$\Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow \quad dx = \frac{1}{2}\, du$

Rewrite the integral with the substitutions

$$\frac{1}{2}\int_{x=\frac{\pi}{8}}^{x=\frac{\pi}{4}} \operatorname{cosec}(u)\cot(u)\, du$$

$f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$ ← Formula booklet

$\Rightarrow \int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + c$

$$\frac{1}{2}\int_{x=\frac{\pi}{8}}^{x=\frac{\pi}{4}} \operatorname{cosec}(u)\cot(u)\, du = \frac{1}{2}\Big[-\operatorname{cosec}(u)\Big]_{x=\frac{\pi}{8}}^{x=\frac{\pi}{4}}$$

Convert the integral limits

$u = 2x \qquad$ when $\quad x = \frac{\pi}{4} \Rightarrow u = \frac{\pi}{2}$

$x = \frac{\pi}{8} \Rightarrow u = \frac{\pi}{4}$

Evaluate the integral with the converted limits

$$\frac{1}{2}\Big[-\operatorname{cosec}(u)\Big]_{u=\frac{\pi}{4}}^{u=\frac{\pi}{2}} = \frac{1}{2}\left[-\operatorname{cosec}\left(\frac{\pi}{2}\right)\right] - \frac{1}{2}\left[-\operatorname{cosec}\left(\frac{\pi}{4}\right)\right]$$

$$= -\frac{1}{2\sin\left(\frac{\pi}{2}\right)} + \frac{1}{2\sin\left(\frac{\pi}{4}\right)}$$

Exact trig values:

$\sin\frac{\pi}{2} = 1$

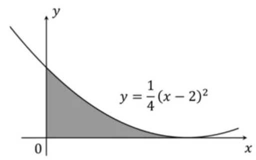
$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$= -\frac{1}{2} + \frac{\sqrt{2}}{2}$$

$$\boxed{\frac{\sqrt{2}-1}{2} \text{ units}^2}$$

Question 10

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.



(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

(i) by calculating it as an area between the curve and the x-axis

(ii) by calculating it as an area between the curve and the y-axis

[6]

(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(a) (i) Find $(0, y)$, $(x, 0)$

When $x = 0 \Rightarrow y = \frac{1}{4}(0-2)^2$

$y = 1$ $\boxed{y\text{-intercept}: (0, 1)}$

When $y = 0 \Rightarrow 0 = \frac{1}{4}(x-2)^2$

$x = 2$ $\boxed{x\text{-intercept}: (2, 0)}$

(ii) Rearrange $y = \frac{1}{4}(x-2)^2$ to make x the subject

$y = \frac{1}{4}(x-2)^2$

$4y = (x-2)^2$

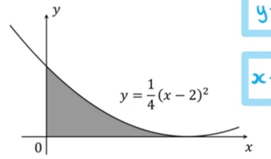
$\pm 2\sqrt{y} = x - 2 \Rightarrow x = 2 \pm 2\sqrt{y}$

The shaded region occurs when $x \leq 2$
$\therefore$ $x = 2 - 2\sqrt{y}$

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.

y-intercept : (0, 1)

x-intercept : (2, 0)



(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

(i) by calculating it as an area between the curve and the x-axis

(ii) by calculating it as an area between the curve and the y-axis

[6]

(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(b) (i) $\int_0^2 \frac{1}{4}(x-2)^2\,dx$

Let $u = x - 2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

Rewrite the integral with the substitutions

$$\frac{1}{4}\int_{x=0}^{x=2} (u)^2\,du$$

Convert the limits of the integral

$u = x - 2$, when $x = 2 \Rightarrow u = 0$

when $x = 0 \Rightarrow u = -2$

Rewrite the integral with the converted limits and evaluate

$$\frac{1}{4}\int_{u=-2}^{u=0} (u)^2\,du = \frac{1}{4}\left[\frac{1}{3}u^3\right]_{u=-2}^{u=0} = \frac{1}{4}\left[\frac{1}{3}(0)^3\right] - \frac{1}{4}\left[\frac{1}{3}(-2)^3\right]$$

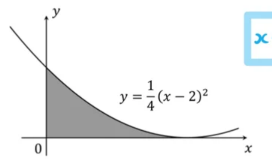
$$= 0 + \frac{8}{12} \qquad \boxed{\frac{2}{3}\text{ units}^2}$$

(ii) $$\int_{y=0}^{y=1} 2 - 2\sqrt{y}\,dy = \left[2y - \frac{4}{3}y^{3/2}\right]_{y=0}^{y=1}$$

$$= \left[2(1) - \frac{4}{3}(1)^{3/2}\right] - \left[2(0) - \frac{4}{3}(0)^{3/2}\right]$$

$$= 2 - \frac{4}{3} \qquad \boxed{\frac{2}{3}\text{ units}^2}$$

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.



x-intercept : $(2, 0)$

(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

(i) by calculating it as an area between the curve and the x-axis

(ii) by calculating it as an area between the curve and the y-axis

[6]

(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(c) $V = \int_a^b \pi y^2 \, dx$ ← Formula booklet

$$V = \int_0^2 \pi\left(\frac{1}{4}(x-2)^2\right)^2 dx = \frac{\pi}{16}\int_0^2 (x-2)^4 \, dx$$

Let $u = x - 2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

Rewrite the integral with the substitutions

$$\frac{\pi}{16}\int_{x=0}^{x=2} (u)^4 \, du$$

Convert the limits of the integral

$u = x - 2$ when $x = 2 \Rightarrow u = 0$

$x = 0 \Rightarrow u = -2$

Rewrite the integral with the converted limits and evaluate

$$\frac{\pi}{16}\int_{u=-2}^{u=0} (u)^4 \, du = \frac{\pi}{16}\left[\frac{u^5}{5}\right]_{u=-2}^{u=0}$$

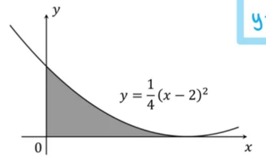
$$= \frac{\pi}{16}\left[\frac{(0)^5}{5}\right] - \frac{\pi}{16}\left[\frac{(-2)^5}{5}\right]$$

$$= 0 + \frac{2}{5}\pi$$

$\boxed{\frac{2}{5}\pi \text{ units}^3}$

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.

y-intercept : (0, 1)



(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

(i) by calculating it as an area between the curve and the x-axis

(ii) by calculating it as an area between the curve and the y-axis

[6]

(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(d) $V = \int_a^b \pi x^2 \, dy$ ← Formula booklet

$$V = \int_0^1 \pi \left(2 - 2\sqrt{y}\right)^2 dy$$

$$= \pi \int_0^1 \left(2 - 2y^{1/2}\right)^2 dy = \pi \int_0^1 4 - 8y^{1/2} + 4y \, dy$$

$$= \pi \left[4y - \frac{16}{3}y^{3/2} + 2y^2\right]_0^1$$

$$= \pi \left[\left(4(1) - \frac{16}{3}(1)^{3/2} + 2(1)^2\right) - (0)\right]$$

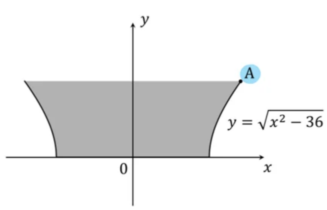
$$= \pi \left(4 - \frac{16}{3} + 2\right)$$

$$\boxed{\frac{2}{3}\pi \text{ units}^3}$$

Note: The volume when the curve is rotated around the y-axis is not necessarily the same as the volume when the curve is rotated around the x-axis

Question 11

The diagram below shows the cross-section of a bowl that a company is planning to begin producing.



As indicated on the diagram, one of the sides of the bowl in the cross-section may be described by the curve $y = \sqrt{x^2 - 36}$, where units for x and y are centimetres. The cross-section is entirely symmetrical about the y-axis. The flat circular bottom of the bowl has a diameter of 12 cm, and the vertical depth of the bowl is 6 cm. For purposes of answering this question, the thickness of the bottom and sides of the bowl may be regarded as negligible.

(a) Find the exact coordinates of the point marked A on the diagram.

[3]

(b) Show that the capacity of the bowl in cm³ is given by

$$\pi \int_0^b (y^2 + 36) \, dy$$

where b is a constant to be determined.

[4]

(c) Hence find the capacity of the bowl.

[2]

(a) Find x when $y = 6$

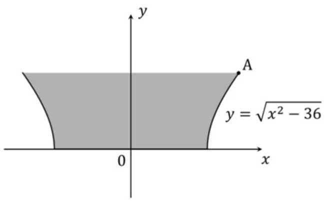
$$6 = \sqrt{x^2 - 36}$$

$$x = \pm\sqrt{72} = \pm 6\sqrt{2}$$

$$\boxed{A\left(6\sqrt{2}, 6\right)}$$

EXAM PAPERS PRACTICE

The diagram below shows the cross-section of a bowl that a company is planning to begin producing.



As indicated on the diagram, one of the sides of the bowl in the cross-section may be described by the curve $y = \sqrt{x^2 - 36}$, where units for x and y are centimetres. The cross-section is entirely symmetrical about the y-axis. The flat circular bottom of the bowl has a diameter of 12 cm, and the vertical depth of the bowl is 6 cm. For purposes of answering this question, the thickness of the bottom and sides of the bowl may be regarded as negligible.

(a) Find the exact coordinates of the point marked A on the diagram.

[3]

(b) Show that the capacity of the bowl in cm³ is given by

$$\pi \int_0^b (y^2 + 36)\, dy$$

where b is a constant to be determined.

[4]

(c) Hence find the capacity of the bowl.

[2]

(b) The capacity of the bowl is found by rotating the curve on one side of the y axis by 2π radians about the y-axis, $\therefore$ the upper limit of the integral will be the height of the bowl

$b = 6$

Write x^2 in terms of y

$y = \sqrt{x^2 - 36}$

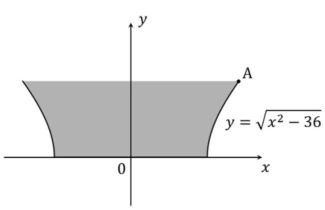
$\Rightarrow x^2 = y^2 + 36$ ⟵ Rearrange for x^2 rather than x as it appears in the formula

$V = \int_a^b \pi x^2 \, dy$ ⟵ formula booklet

$V = \int_0^6 \pi (y^2 + 36)\, dy$

$$\boxed{V = \pi \int_0^6 (y^2 + 36)\, dy}$$

The diagram below shows the cross-section of a bowl that a company is planning to begin producing.



As indicated on the diagram, one of the sides of the bowl in the cross-section may be described by the curve $y = \sqrt{x^2 - 36}$, where units for x and y are centimetres. The cross-section is entirely symmetrical about the y-axis. The flat circular bottom of the bowl has a diameter of 12 cm, and the vertical depth of the bowl is 6 cm. For purposes of answering this question, the thickness of the bottom and sides of the bowl may be regarded as negligible.

(a) Find the exact coordinates of the point marked A on the diagram.

[3]

(b) Show that the capacity of the bowl in cm³ is given by

$$\pi \int_0^b (y^2 + 36)\, dy$$

where b is a constant to be determined.

[4]

$$\boxed{V = \pi \int_0^6 (y^2 + 36)\, dy}$$

(c) Hence find the capacity of the bowl.

[2]

(c) $V = \pi \left[\frac{1}{3} y^3 + 36y \right]_0^6$

$= \pi \left[\left(\frac{1}{3}(6)^3 + 36(6) \right) - \left(\frac{1}{3}(0)^3 + 36(0) \right) \right]$

$= \pi (288 - 0)$

$= 288\pi$

$= 904.778684\ldots$

$$\boxed{V = 905 \text{ cm}^3 \ (3\text{ sf})}$$