Question 1

Let $f(x) = 3x^2$.

By differentiating from first principles, show that $f'(x) = 6x$.

[4]

$f'(x)$ from first principles

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (in formula booklet)

$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$

$f'(x) = \lim_{h \to 0} \frac{\cancel{3x^2} + 6hx + 3h^2 - \cancel{3x^2}}{h}$

$f'(x) = \lim_{h \to 0} \frac{6hx + 3h^2}{h} = \lim_{h \to 0} (6x + 3h)$

$\boxed{f'(x) = 6x}$

Question 2

Let $f(x) = \sin x$.

(a) Solve the equation $f'(x) = f'''(x)$ in the interval $0 \le x \le 2\pi$.

[4]

(b) Show that $f^{(4)}(x) = f(x)$.

[2]

a) $f(x) = \sin x \rightarrow f'(x) = \cos x$

$\rightarrow f''(x) = -\sin x \rightarrow f'''(x) = -\cos x$

$\therefore \cos x = -\cos x \rightarrow 2\cos x = 0$

$\cos x = 0$

$\boxed{x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}}$

Let $f(x) = \sin x$.

(a) Solve the equation $f'(x) = f'''(x)$ in the interval $0 \le x \le 2\pi$.

[4]

(b) Show that $f^{(4)}(x) = f(x)$.

[2]

b) $f^{(4)}(x) = f''''(x)$

$f'''(x) = -\cos x \rightarrow f^{(4)}(x) = -(-\sin x) = \sin x$

$\boxed{\therefore f^{(4)}(x) = f(x) = \sin x}$

Question 3

Find the derivative of each of the following functions:

(a) $f(x) = \cot\left(x + \frac{\pi}{3}\right)$ [2]

(b) $g(x) = 5^x - 3\log_3 x$ [2]

(c) $h(x) = \arcsin 4x$ [3]

a) $f(x) = \cot x \rightarrow f'(x) = -\operatorname{cosec}^2 x$ (in formula booklet)

$$\boxed{\therefore f'(x) = -\operatorname{cosec}^2\left(x + \frac{\pi}{3}\right)}$$

Find the derivative of each of the following functions:

(a) $f(x) = \cot\left(x + \frac{\pi}{3}\right)$ [2]

(b) $g(x) = 5^x - 3\log_3 x$ [2]

(c) $h(x) = \arcsin 4x$ [3]

b) $f(x) = a^x \rightarrow f'(x) = a^x(\ln a)$ (in formula booklet)

$f(x) = \log_a x \rightarrow f'(x) = \frac{1}{x \ln a}$ (in formula booklet)

$$\boxed{g'(x) = 5^x(\ln 5) - \frac{3}{x\ln 3}}$$

Find the derivative of each of the following functions:

(a) $f(x) = \cot\left(x + \frac{\pi}{3}\right)$ [2]

(b) $g(x) = 5^x - 3\log_3 x$ [2]

(c) $h(x) = \arcsin 4x$ [3]

c) $f(x) = \arcsin x \rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$ (in formula booklet)

Chain rule: $y = g(u)$, where $u = f(x)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (in formula booklet)

$h(x) = \arcsin u$, $u = 4x \rightarrow \frac{du}{dx} = 4$

$h'(x) = \frac{1}{\sqrt{1-u^2}} \times 4 = \frac{4}{\sqrt{1-(4x)^2}}$

$$\boxed{h'(x) = \frac{4}{\sqrt{1-16x^2}}}$$

EXAM PAPERS PRACTICE

Question 4

(a) For the curve defined by $y = \tan 3x$, show that

$$\frac{d^2y}{dx^2} = 18(\tan 3x + \tan^3 3x)$$

[4]

(b) For the curve defined by $y = \arctan x$, show that

$$y'' = -\frac{2x}{x^4 + 2x^2 + 1}$$

[4]

a) $y = \tan x \rightarrow \frac{dy}{dx} = \sec^2 x$ (in formula booklet)

Chain rule: $y = g(u)$, where $u = f(x)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (in formula booklet)

$y = \tan u$, $u = 3x \rightarrow \frac{du}{dx} = 3$

$\frac{dy}{dx} = \sec^2 u \times 3 = 3\sec^2 3x$ $(= 3(\sec 3x)^2)$

Here we have to apply the chain rule twice.

$\frac{dy}{dx} = 3\sec^2 3x$, $v = \sec 3x$, $\frac{dv}{dx} = 3\sec 3x \tan 3x$

$\frac{d^2y}{dx^2} = 6\sec 3x(3\sec 3x \tan 3x) = 18\sec^2 3x \tan 3x$

$\frac{d^2y}{dx^2} = 18(1 + \tan^2 3x)\tan 3x$ trig identity $\sec^2 x = 1 + \tan^2 x$ (in formula booklet)

$$\boxed{\frac{d^2y}{dx^2} = 18(\tan 3x + \tan^3 3x)}$$

(a) For the curve defined by $y = \tan 3x$, show that

$$\frac{d^2y}{dx^2} = 18(\tan 3x + \tan^3 3x)$$

[4]

(b) For the curve defined by $y = \arctan x$, show that

$$y'' = -\frac{2x}{x^4 + 2x^2 + 1}$$

[4]

b) $y = \arctan x \rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$ (in formula booklet)

$\therefore y' = \frac{1}{1+x^2} = (1+x^2)^{-1}$

Chain rule: $y = g(u)$, where $u = f(x)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (in formula booklet)

$y' = u^{-1}$, $u = 1 + x^2$, $u' = 2x$

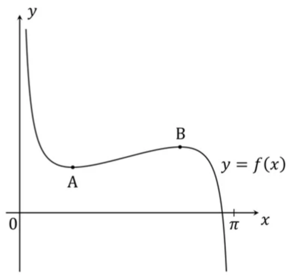
$y'' = -(1+x^2)^{-2}(2x) = -\frac{2x}{(1+x^2)^2} = -\frac{2x}{(1+x^2)(1+x^2)}$

$$\boxed{y'' = -\frac{2x}{x^4 + 2x^2 + 1}}$$

Question 5

Consider the function f defined by $f(x) = 2x + \cot x,\ 0 < x < \pi$.

The following diagram shows the graph of the curve $y = f(x)$:



The points marked A and B are the turning points of the graph.

(a) (i) Find $f'(x)$.

(ii) Hence find the coordinates of points A and B.

[6]

(b) Find the equation of the normal to the graph at the point where the x-coordinate is equal to $\frac{\pi}{2}$.

[4]

a) i) $f(x) = \cot x \longrightarrow f'(x) = -\operatorname{cosec}^2 x$ (in formula booklet)

$f(x) = 2x + \cot x$

$\longrightarrow \boxed{f'(x) = 2 - \operatorname{cosec}^2 x}$

ii) solve $f'(x) = 0$

$2 - \operatorname{cosec}^2 x = 0 \longrightarrow 2 = \operatorname{cosec}^2 x = \frac{1}{\sin^2 x}$

$\sin^2 x = \frac{1}{2} \longrightarrow \sin x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ $(0 < x < \pi)$

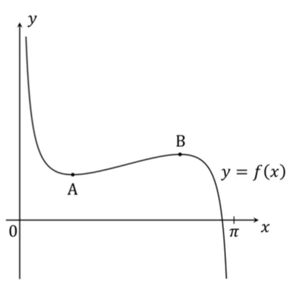
$f\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + 1$

$f\left(\frac{3\pi}{4}\right) = 2\left(\frac{3\pi}{4}\right) + \cot\left(\frac{3\pi}{4}\right) = \frac{3\pi}{2} - 1$

$\boxed{A\left(\frac{\pi}{4}, \frac{\pi}{2} + 1\right) \text{ and } B\left(\frac{3\pi}{4}, \frac{3\pi}{2} - 1\right)}$

Consider the function f defined by $f(x) = 2x + \cot x,\ 0 < x < \pi$.

The following diagram shows the graph of the curve $y = f(x)$:



The points marked A and B are the turning points of the graph.

(a) (i) Find $f'(x)$. $\boxed{f'(x) = 2 - \operatorname{cosec}^2 x}$

(ii) Hence find the coordinates of points A and B.

[6]

(b) Find the equation of the normal to the graph at the point where the x-coordinate is equal to $\frac{\pi}{2}$.

[4]

b) $f\left(\frac{\pi}{2}\right) = \cancel{2}\left(\frac{\pi}{\cancel{2}}\right) + \cot\left(\frac{\pi}{2}\right) = \pi + 0$ $\therefore$ pt $\left(\frac{\pi}{2}, \pi\right)$

$f'\left(\frac{\pi}{2}\right) = 2 - \operatorname{cosec}^2\left(\frac{\pi}{2}\right) = 2 - 1$ $\therefore m_T = 1 \rightarrow m_N = -1$

$y - y_1 = m(x - x_1)$

$y - \pi = -\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$

$\boxed{y = -x + \frac{3\pi}{2}}$

Question 6

For each of the following, find $\frac{dy}{dx}$ by differentiating implicitly with respect to x.

(a) $x^2 + y^2 = 16$ [2]

(b) $4x^2 - 3x = y^2 + 2y$ [2]

(c) $\frac{(x+y)^2}{3x} = 1$ [3]

(d) $\sqrt{x^2 + y^3} = 4$ [4]

a) $x^2 + y^2 = 16$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$

$2x + 2y\frac{dy}{dx} = 0$

$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$

For each of the following, find $\frac{dy}{dx}$ by differentiating implicitly with respect to x.

(a) $x^2 + y^2 = 16$ [2]

(b) $4x^2 - 3x = y^2 + 2y$ [2]

(c) $\frac{(x+y)^2}{3x} = 1$ [3]

(d) $\sqrt{x^2 + y^3} = 4$ [4]

b) $4x^2 - 3x = y^2 + 2y$

$\frac{d}{dx}(4x^2) - \frac{d}{dx}(3x) = \frac{d}{dx}(y^2) + \frac{d}{dx}(2y)$

$8x - 3 = 2y\frac{dy}{dx} + 2\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{8x - 3}{2y + 2}}$

For each of the following, find $\frac{dy}{dx}$ by differentiating implicitly with respect to x.

(a) $x^2 + y^2 = 16$ [2]

(b) $4x^2 - 3x = y^2 + 2y$ [2]

(c) $\frac{(x+y)^2}{3x} = 1$ [3]

(d) $\sqrt{x^2 + y^3} = 4$ [4]

c) $\frac{(x+y)^2}{3x} = 1 \longrightarrow (x+y)^2 = 3x$

$x^2 + 2xy + y^2 = 3x$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(3x)$

$2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 3$

$$\boxed{\frac{dy}{dx} = \frac{3 - 2x - 2y}{2x + 2y}}$$

For each of the following, find $\frac{dy}{dx}$ by differentiating implicitly with respect to x.

(a) $x^2 + y^2 = 16$ [2]

(b) $4x^2 - 3x = y^2 + 2y$ [2]

(c) $\frac{(x+y)^2}{3x} = 1$ [3]

(d) $\sqrt{x^2 + y^3} = 4$ [4]

d) $\sqrt{x^2 + y^3} = 4 = (x^2 + y^3)^{\frac{1}{2}} = 4$

$\frac{1}{2}(x^2 + y^3)^{-\frac{1}{2}}\left(\frac{d}{dx}(x^2) + \frac{d}{dx}(y^3)\right) = \frac{d}{dx}(4)$

$\frac{1}{2}(x^2 + y^3)^{-\frac{1}{2}}\left(2x + 3y^2\frac{dy}{dx}\right) = 0$

$3y^2\frac{dy}{dx}(x^2 + y^3)^{-\frac{1}{2}} = -2x(x^2 + y^3)^{-\frac{1}{2}}$

$$\boxed{\frac{dy}{dx} = -\frac{2x}{3y^2}}$$

EXAM PAPERS PRACTICE

Question 7

A curve is described by the equation

$$\frac{2}{x} - \frac{1}{y} = 1$$

(a) Use implicit differentiation with respect to x to show that

$$\frac{dy}{dx} = \frac{2y^2}{x^2}$$

[2]

(b) Use your result from part (a) to find the equation of the

(i) tangent

(ii) normal

to the curve at the point $(1, 1)$.

[4]

(c) (i) Rearrange the equation of the curve into the form $y = f(x)$.

(ii) Hence find an expression for $\frac{dy}{dx}$ entirely in terms of x.

[5]

(d) Verify that your answer to part (c)(ii) and the result from part (a) both give the same value for the gradient of the tangent to the curve at the point $(1, 1)$.

[2]

a) $\frac{2}{x} - \frac{1}{y} = 1 \longrightarrow 2x^{-1} - y^{-1} = 1$

$\frac{d}{dx}(2x^{-1}) - \frac{d}{dx}(y^{-1}) = \frac{d}{dx}(1)$

$-2x^{-2} + y^{-2}\frac{dy}{dx} = -\frac{2}{x^2} + \frac{1}{y^2}\frac{dy}{dx} = 0$

$\frac{1}{y^2}\frac{dy}{dx} = \frac{2}{x^2}$

$\boxed{\frac{dy}{dx} = \frac{2y^2}{x^2}}$

A curve is described by the equation

$$\frac{2}{x} - \frac{1}{y} = 1$$

(a) Use implicit differentiation with respect to x to show that

$$\frac{dy}{dx} = \frac{2y^2}{x^2}$$

[2]

(b) Use your result from part (a) to find the equation of the

(i) tangent

(ii) normal

to the curve at the point $(1, 1)$.

[4]

(c) (i) Rearrange the equation of the curve into the form $y = f(x)$.

(ii) Hence find an expression for $\frac{dy}{dx}$ entirely in terms of x.

[5]

(d) Verify that your answer to part (c)(ii) and the result from part (a) both give the same value for the gradient of the tangent to the curve at the point $(1, 1)$.

[2]

b) sub in $x = 1$ and $y = 1$ into $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2(1)^2}{(1)^2} = 2 \quad \therefore m_T = 2 \rightarrow m_N = -\frac{1}{2}$

$y - y_1 = m(x - x_1)$ (in formula booklet)

i) tangent: $y - 1 = 2(x - 1)$

$\boxed{y = 2x - 1}$

ii) normal: $y - 1 = -\frac{1}{2}(x - 1)$

$\boxed{y = -\frac{1}{2}x + \frac{3}{2}}$

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A curve is described by the equation

$$\frac{2}{x} - \frac{1}{y} = 1$$

(a) Use implicit differentiation with respect to x to show that

$$\frac{dy}{dx} = \frac{2y^2}{x^2}$$

[2]

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[4]

(c) (i) Rearrange the equation of the curve into the form $y = f(x)$.

(ii) Hence find an expression for $\frac{dy}{dx}$ entirely in terms of x.

[5]

(d) Verify that your answer to part (c)(ii) and the result from part (a) both give the same value for the gradient of the tangent to the curve at the point $(1, 1)$.

[2]

c) i) $\frac{2}{x} - \frac{1}{y} = 1 \longrightarrow \frac{1}{y} = \frac{2}{x} - 1 = \frac{2-x}{x}$

$$\boxed{y = \frac{x}{2-x}}$$

ii) Quotient rule (in formula booklet)

$$y = \frac{u}{v} \longrightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = x$, $u' = 1$ $v = 2-x$, $v' = -1$

$$\frac{dy}{dx} = \frac{(2-x)(1) - x(-1)}{(2-x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2}{(2-x)^2}}$$

A curve is described by the equation

$$\frac{2}{x} - \frac{1}{y} = 1$$

(a) Use implicit differentiation with respect to x to show that

$$\frac{dy}{dx} = \frac{2y^2}{x^2}$$

[2]

(b) Use your result from part (a) to find the equation of the

(i) tangent

(ii) normal

to the curve at the point $(1, 1)$.

[4]

(c) (i) Rearrange the equation of the curve into the form $y = f(x)$.

(ii) Hence find an expression for $\frac{dy}{dx}$ entirely in terms of x. $\boxed{\frac{dy}{dx} = \frac{2}{(2-x)^2}}$

[5]

(d) Verify that your answer to part (c)(ii) and the result from part (b)(i) both give the same value for the gradient of the tangent to the curve at the point $(1, 1)$.

[2]

d) sub $x = 1$ into $\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{2}{(2-1)^2} = 2\text{, which is the same result found in part (b).}}$

Question 8

An international mission has landed a rover on the planet Mars. After landing, the rover deploys a small drone on the surface of the planet, then rolls away to a distance of 6 metres in order to observe the drone as it lifts off into the air. Once the rover has finished moving away, the drone ascends vertically into the air at a constant speed of 2 metres per second.

Let D be the distance, in metres, between the rover and the drone at time t seconds. Let h be the height, in metres, of the drone above the ground at time t seconds. The entire area where the rover and drone are situated may be assumed to be perfectly horizontal.

(a) Show that

$$D = \sqrt{h^2 + 36}$$

[2]

(b) (i) Explain why $\frac{\mathrm{d}h}{\mathrm{d}t} = 2$.

(ii) Hence use implicit differentiation to show that

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{2h}{\sqrt{h^2 + 36}}$$

[5]

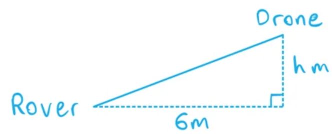
(c) Find

(i) the rate at which the distance between the rover and the drone is increasing at the moment when the drone is 8 metres above the ground.

(ii) the height of the drone above the ground at the moment when the distance between the rover and the drone is increasing at a rate of 1 ms^{-1}.

[4]

a) This is just Pythagoras.



$$\boxed{D = \sqrt{h^2 + 6^2} = \sqrt{h^2 + 36}}$$

An international mission has landed a rover on the planet Mars. After landing, the rover deploys a small drone on the surface of the planet, then rolls away to a distance of 6 metres in order to observe the drone as it lifts off into the air. Once the rover has finished moving away, the drone ascends vertically into the air at a constant speed of 2 metres per second.

Let D be the distance, in metres, between the rover and the drone at time t seconds. Let h be the height, in metres, of the drone above the ground at time t seconds. The entire area where the rover and drone are situated may be assumed to be perfectly horizontal.

(a) Show that

$$D = \sqrt{h^2 + 36}$$

[2]

(b) (i) Explain why $\frac{\mathrm{d}h}{\mathrm{d}t} = 2$.

(ii) Hence use implicit differentiation to show that

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{2h}{\sqrt{h^2 + 36}}$$

[5]

(c) Find

(i) the rate at which the distance between the rover and the drone is increasing at the moment when the drone is 8 metres above the ground.

(ii) the height of the drone above the ground at the moment when the distance between the rover and the drone is increasing at a rate of 1 ms^{-1}.

[4]

b) i) $\frac{dh}{dt}$ is the vertical speed in ms^{-1}, and the question says this is 2ms^{-1}

ii) $D = \sqrt{h^2 + 36} = (h^2 + 36)^{1/2}$

$$\frac{dD}{dt} = \frac{1}{2}(h^2 + 36)^{-1/2}\left(2h\frac{dh}{dt}\right) = \frac{h}{\sqrt{h^2 + 36}}\frac{dh}{dt}$$

$$\boxed{\frac{dD}{dt} = \frac{2h}{\sqrt{h^2 + 36}}}$$

An international mission has landed a rover on the planet Mars. After landing, the rover deploys a small drone on the surface of the planet, then rolls away to a distance of 6 metres in order to observe the drone as it lifts off into the air. Once the rover has finished moving away, the drone ascends vertically into the air at a constant speed of 2 metres per second.

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(a) Show that

$$D = \sqrt{h^2 + 36}$$

[2]

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(ii) Hence use implicit differentiation to show that

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{2h}{\sqrt{h^2 + 36}}$$

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(i) the rate at which the distance between the rover and the drone is increasing at the moment when the drone is 8 metres above the ground.

(ii) the height of the drone above the ground at the moment when the distance between the rover and the drone is increasing at a rate of 1 ms^{-1}.

[4]

c) i) sub $h = 8$ into $\frac{dD}{dt}$

$$\frac{dD}{dt} = \frac{2(8)}{\sqrt{(8)^2 + 36}} = \frac{16}{10}$$

$$\boxed{\frac{dD}{dt} = 1.6 \text{ ms}^{-1}}$$

ii) rearrange $\frac{dD}{dt}$ for h

$$\frac{dD}{dt} = \frac{2h}{\sqrt{h^2 + 36}} \longrightarrow \left(\frac{dD}{dt}\right)^2 = \frac{4h^2}{h^2 + 36}$$

$$\left(\frac{dD}{dt}\right)^2 (h^2 + 36) = h^2 \left(\frac{dD}{dt}\right)^2 + 36 \left(\frac{dD}{dt}\right)^2 = 4h^2$$

$$4h^2 - h^2 \left(\frac{dD}{dt}\right)^2 = 36 \left(\frac{dD}{dt}\right)^2$$

$$h^2 \left(4 - \left(\frac{dD}{dt}\right)^2\right) = 36 \left(\frac{dD}{dt}\right)^2$$

$$h = \sqrt{\frac{36 \left(\frac{dD}{dt}\right)^2}{4 - \left(\frac{dD}{dt}\right)^2}}$$

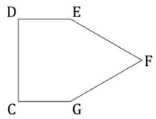
sub in $\frac{dD}{dt} = 1$

$$h = \sqrt{\frac{36(1)^2}{4 - (1)}} = \sqrt{12} = 2\sqrt{3}$$

$$\boxed{h = 3.464\ldots = 3.46 \text{ m } (3 \text{ s.f.})}$$

Question 9

In the diagram below, CDEFG is the outline of a type of informational signboard that a county council plans to use in one of its parks. The shape is formed by a rectangle CDEG, to one side of which an equilateral triangle EFG has been appended.



The signboards will be produced in various different sizes. However because of the cost of the edging that must go around the perimeter of the signboards, the council is eager to design the signboards so that the area of a signboard is the maximum possible for a given perimeter.

Let $|CD| = x$ cm and let $|DE| = y$ cm.

(a) (i) Write down an expression in terms of x and y for the perimeter of the signboard, P.

(ii) Hence use implicit differentiation to find $\frac{dP}{dx}$.

[3]

(b) Explain why, for a given perimeter, it must be true that $\frac{dP}{dx} = 0$, and use this fact to show that $\frac{dy}{dx} = -\frac{3}{2}$.

[3]

(c) Show that the area, A, of the signboard is given by $A = xy + \frac{\sqrt{3}}{4}x^2$.

[4]

(d) Hence use implicit differentiation to find the ratio of y to x that gives the maximum area.

[5]

a) i) $CD = EF = FG = x$ (equilateral triangle)

$$\boxed{P = 3x + 2y}$$

ii) $$\boxed{\frac{dP}{dx} = 3 + 2\frac{dy}{dx}}$$

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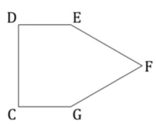

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$$\boxed{\frac{dP}{dx} = 3 + 2\frac{dy}{dx}}$$

[3]

(b) Explain why, for a given perimeter, it must be true that $\frac{dP}{dx} = 0$, and use this fact to show that $\frac{dy}{dx} = -\frac{3}{2}$.

[3]

(c) Show that the area, A, of the signboard is given by $A = xy + \frac{\sqrt{3}}{4}x^2$.

[4]

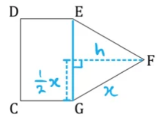
(d) Hence use implicit differentiation to find the ratio of y to x that gives the maximum area.

[5]

b) If P is given, then P must be fixed - ie. it doesn't change as x changes.

$\therefore \frac{dP}{dx} = 3 + 2\frac{dy}{dx} = 0 \longrightarrow \frac{dy}{dx} = -\frac{3}{2}$

In the diagram below, CDEFG is the outline of a type of informational signboard that a county council plans to use in one of its parks. The shape is formed by a rectangle CDEG, to one side of which an equilateral triangle EFG has been appended.



The signboards will be produced in various different sizes. However because of the cost of the edging that must go around the perimeter of the signboards, the council is eager to design the signboards so that the area of a signboard is the maximum possible for a given perimeter.

Let $|CD| = x$ cm and let $|DE| = y$ cm.

(a) (i) Write down an expression in terms of x and y for the perimeter of the signboard, P.

(ii) Hence use implicit differentiation to find $\frac{dP}{dx}$.

[3]

(b) Explain why, for a given perimeter, it must be true that $\frac{dP}{dx} = 0$, and use this fact to show that $\frac{dy}{dx} = -\frac{3}{2}$.

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[5]

c) Area, A = rectangle + triangle

rectangle = xy

triangle = $\frac{1}{2}(x)\left(\sqrt{x^2 - \left(\frac{1}{2}x\right)^2}\right) = \frac{\sqrt{3}}{4}x^2$

$$\boxed{\therefore A = xy + \frac{\sqrt{3}}{4}x^2}$$

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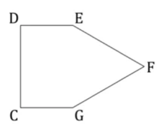

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d) Differentiate implicitly with respect to x and use $\frac{dy}{dx} = -\frac{3}{2}$

$$\frac{dA}{dx} = y + x\frac{dy}{dx} + \frac{\sqrt{3}}{2}x = y - \frac{3}{2}x + \frac{\sqrt{3}}{2}x$$

$\frac{dA}{dx} = 0$ for a maximum

$$y - \frac{3}{2}x + \frac{\sqrt{3}}{2}x = 0$$

$$\therefore y = \frac{3 - \sqrt{3}}{2}x \quad \text{or} \quad x = \frac{2}{3 - \sqrt{3}}y$$

$\therefore$ the ratio of $y : x$ is

$$y : x = 1 : \frac{2}{3 - \sqrt{3}} = 1 : 1.577\ldots$$

$$\boxed{y : x = 1 : 1.58 \ (3sf)}$$