EXAM PAPERS PRACTICE

Pearson Edexcel

Mark Scheme

Summer 2023

Pearson Edexcel GCE
In A Level Further Mathematics (9FM0)
Paper 4A Further Pure Mathematics 2

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Summer 2023
Publications Code 9FM0_4A_2206_MS

General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EXAM PAPERS PRACTICE

EDEXCEL GCE MATHEMATICS
General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol √ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- ∗ The answer is printed on the paper
- □ The second mark is dependent on gaining the first mark

4. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

5. Where a candidate has made multiple responses <u>and indicates which response they wish to submit</u>, examiners should mark this response.

If there are several attempts at a question which have not been crossed out, examiners should mark the final answer which is the answer that is the most complete.

6. Ignore wrong working or incorrect statements following a correct answer.

7. Mark schemes will firstly show the solution judged to be the most common response expected from candidates. Where appropriate, alternatives answers are provided in the notes. If examiners are not sure if an answer is acceptable, they will check the mark scheme to see if an alternative answer is given for the method used.

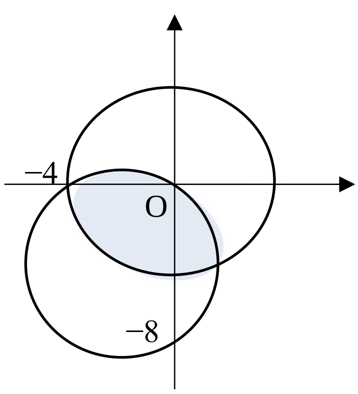
Question	Scheme	Marks	AOs
1(a)	$\det\begin{pmatrix}-1-\lambda & a\\ 3 & 8-\lambda\end{pmatrix}=0\Rightarrow(-1-\lambda)(8-\lambda)-3a=0\Rightarrow...$	M1	1.1b
	$\lambda^2-7\lambda-3a-8=0$ o.e.	A1	1.1b
		(2)	
(b)	$\mathbf{A}^2-7\mathbf{A}-3a\mathbf{I}-8\mathbf{I}=0\Rightarrow\mathbf{A}^3=7\mathbf{A}^2+(3a+8)\mathbf{A}$	M1	1.1b
	$\Rightarrow\mathbf{A}^3=7(7\mathbf{A}+(3a+8)\mathbf{I})+(3a+8)\mathbf{A}$ Or $\mathbf{A}^3=7\begin{pmatrix}-1 & a\\ 3 & 8\end{pmatrix}^2+(8+3a)\begin{pmatrix}-1 & a\\ 3 & 8\end{pmatrix}$ $=7\begin{pmatrix}1+3a & 7a\\ 21 & 3a+64\end{pmatrix}+\begin{pmatrix}-8-3a & 8a+3a^2\\ 24+9a & 64+24a\end{pmatrix}$	M1	2.1
	$\Rightarrow\mathbf{A}^3=(3a+8+49)\mathbf{A}+7(3a+8)\mathbf{I}\Rightarrow 3a+57=1\Rightarrow a=...$ Or $\begin{pmatrix}b-1 & a\\ 3 & b+8\end{pmatrix}=\begin{pmatrix}18a-1 & 3a^2+57a\\ 171+9a & 512+27a\end{pmatrix}\Rightarrow$ e.g. $3=171+9a\Rightarrow a=...$ Or $\mathbf{A}^3=\begin{pmatrix}-1 & a\\ 3 & 8\end{pmatrix}+\begin{pmatrix}18a & 3a^2+56a\\ 168+9a & 504+27a\end{pmatrix}\Rightarrow$ e.g. $18a=504+27a\Rightarrow a=...$	M1	1.1b
	$\Rightarrow a=-\dfrac{56}{3}, b=-336$	A1	1.1b
		(4)	
		(6 marks)	

Notes:

(a)
M1: Correct method to find the characteristic equation for **A**, condone missing = 0
A1: Correct simplified characteristic equation.

(b)
Note: this questions asks the candidates to use the Cayley-Hamilton theorem so any other approach that doesn't score the first method mark scores no marks
M1: Uses the Cayley-Hamilton theorem with their equation and multiplies though by **A** to find an equation for $\mathbf{A}^3$

M1: Substitutes for $\mathbf{A}^2$ to obtain an equation for $\mathbf{A}^3$ in terms of a, **I** and **A.** Alternatively substitutes in the matrix **A** and attempts to square **M1:** Equates coefficient(s) of **A** to 1 and proceeds to find a value for a. Alternatively equates elements to find a value for a. **A1:** Correct values for a and b. **Special case:** Missing matrix **I** from their working can score maximum of M1 (if multiply by A correctly) M1M1A0 unless implied **I** from their working this can score all marks.

Question	Scheme	Marks	AOs
2 (a)	$\lvert -4-6\rvert = 10$ and $2\lvert -4+3\mathrm{i}\rvert = \left\{2\sqrt{4^2+3^2} = 2\times 5\right\} = 10$ {so -4 is on the locus} Or $\lvert -8\mathrm{i}-6\rvert = \left\{\sqrt{64+36} =\right\} 10$ and $2\lvert -8\mathrm{i}+3\mathrm{i}\rvert \left\{= 2\lvert -5\mathrm{i}\rvert = 2\times 5\right\} = 10$ {so $-8\mathrm{i}$ is on the locus}	M1	1.1b
	$\lvert -4-6\rvert = 10$ and $2\lvert -4+3\mathrm{i}\rvert = \left\{2\sqrt{4^2+3^2} = 2\times 5\right\} = 10$ {so -4 is on the locus} **and** $\lvert -8\mathrm{i}-6\rvert = \left\{\sqrt{64+36} =\right\} 10$ and $2\lvert -8\mathrm{i}+3\mathrm{i}\rvert \left\{= 2\lvert -5\mathrm{i}\rvert = 2\times 5\right\} = 10$ {so $-8\mathrm{i}$ is on the locus} **and** $\lvert -6\rvert = 6$ and $\lvert 3\mathrm{i}\rvert = 6$ {so the origin is also on the locus}	A1	1.1b
		(2)	
	Alternative: Finds the Cartesian equation of the circle, any form $(x-6)^2 + y^2 = 4x^2 + 4(y+3)^2$ $3x^2 + 3y^2 + 24y + 12x = 0$ $x^2 + y^2 + 8y + 4x = 0$ $(x+2)^2 + (y+4)^2 = 20$ Substitutes in either $(-4,0)$ or $(0,-8)$ into the equation to show it holds	M1	1.1b
	Substitutes in $(-4,0)$ **and** $(0,-8)$ **and** $(0,0)$ into the equation to show it holds	A1	1.1b
		(2)	
(b)+(c)	 **(b)** Circle passing through the origin	M1	2.2a
	Correct circle, with axis intercepts	A1	1.1b
		(2)	
	(c) Circle with centre (0,0) and radius 4	B1	1.1b
	Shades area inside both circles.	B1ft	2.5
		(2)	

(6 marks)
Notes:
(a) **M1:** Verifies the equation is satisfied by at least one of the non-zero points, with evidence of correct method of the modulus seen (e.g. implied by correct value). **A1:** All three points correctly checked to be on the locus. **Alternative** **M1:** Finds the Cartesian equation of the circle and verifies that the equation is satisfied by at least one of the non-zero points. **A1:** All three points correctly checked to be on the locus. **Special case**: **M1A0** Candidates shows that the locus of *P* passes through $z=-4-8\text{i}$ $\lvert -10-8\text{i}\rvert = \left\{\sqrt{100+64} = \right\} 2\sqrt{41}$ $2\lvert -4-5\text{i}\rvert = \left\{\sqrt{16+25} = \right\} 2\sqrt{41}$ Cartesian approach substitutes $(-4,-8)$ to show it satisfies the equation
(b) **M1:** Any complete circle passing through the origin. **A1:** Correct circle drawn and labelled. Look for centre in the third quadrant and passing through −4, 0 and −8i.
(c) **B1:** Circle centre at origin and some indication that the radius is 4. Can sketch part of the circle as long as sufficient to show the overlap. **B1ft:** Area inside both ‘circles’ shaded. Allow as long as they have overlapping circles.

Question	Scheme	Marks	AOs
3(a)	Two of • $U_1 = 25$ because there are 25 subscribers at the end of week 1 • 20% of subscribers leave so 80% so $0.8U_n$ remaining subscribers or $U_n - 0.2U_n$ • At end of week n + 1 there are a new 20(n + 1) subscribers added to those from week n	M1	3.3
	All three points above put together in conclusion Hence $U_{n+1} = 0.8U_n + 20(n+1),\ U_1 = 25$	A1	2.4
		(2)	
(b)	$n = 1 \Rightarrow U_1 = 325\times1 + 100\times1 - 400 = 325 - 300 = 25$ {so the result is true for n = 1}	B1	2.2a
	(Assume true for n = k then) $U_{k+1} = 0.8U_k + 20k + 20 = 0.8\left(325\left(\frac{4}{5}\right)^{k-1} + 100k - 400\right) + 20k + 20$	M1	1.1b
	$= 325\times\frac{4}{5}\times\left(\frac{4}{5}\right)^{k-1} + 80k - 320 + 20k + 20$ $= 325\times\left(\frac{4}{5}\right)^{k} + 100k - 300$	M1	1.1b
	$= 325\times\left(\frac{4}{5}\right)^{k} + 100(k+1) - 400$	A1	2.1
	Hence if the result is **true for n = k, then it is true for n = k + 1**, and as it **is true for n = 1**, so it **is true for all positive integers n** or $n \geq 1$	A1	2.4
		(5)	
	Alternative $n = 1 \Rightarrow U_1 = 325\times1 + 100\times1 - 400 = 325 - 300 = 25$ so the result is true for n = 1	B1	2.2a
	$U_{k+1} = 325\left(\frac{4}{5}\right)^{k} + 100(k+1) - 400 = \frac{4}{5}\times325\left(\frac{4}{5}\right)^{k-1} + 100k - 300$	M1	1.1b
	$= \frac{4}{5}\times\left(325\left(\frac{4}{5}\right)^{k-1} + 100k - 400\right) + 20k + 20$	M1	1.1b
	$U_{k+1} = 0.8U_k + 20(k+1)$	A1	2.1
	Hence **if the result is true for n = k, then it is true for n = k + 1**, and as it **is true for n = 1**, so it **is true for all positive integers n** or $n \geq 1$	A1	2.4

		(5)	
(c)	An attempt at either $U_{24} = 325 \times 0.8^{23} + 2400 - 400 = 2001.9...$ Or $U_{25} = 325 \times 0.8^{24} + 2500 - 400 = 2101.5...$ or $U_{26} = 325 \times 0.8^{25} + 2600 - 400 = 2201.2....$ Or $U_{27} = 325 \times 0.8^{26} + 2700 - 400 = 2300.98...$	M1	3.4
	Correct value for their number of weeks 24, 25, 26 or 27. Compares the value after 6 months with 1800 and draws a conclusion e.g. This is overestimating the actual amount by 400 people and therefore not a very good model.	A1	3.5a
		(2)	
			(9 marks)

Notes:

(a)
M1: See scheme. Explains how the assumptions lead to at least two of the aspects indicated. Accept less formal explanations as long as the intent is clear.
A1: All three aspects explained and put together to set up the model.

(b)
B1: Checks the case for n =1 holds.
M1: Makes the inductive assumption (may be implied by working) and substitutes the closed form for U_k into the recurrence relation for U_{k+1} or equivalent work with different variable (e.g. n instead of k) or indexing (e.g. from $k - 1$ to k).

M1: Simplifies to the point of combining the powers of $\frac{4}{5}$ to one term.

A1: For correct work leading to the form shown. The k + 1 must be seen in the added term but allow just k for the power.
A1: For a completely correct proof (all previous marks must be gained) with a conclusion that includes all of the bold statements in the scheme or equivalents.

Alternative
B1: Checks the case for n =1 holds.

M1: Writes out the term U_{k+1} and starts the process to write in terms of U_k by factorising out $\frac{4}{5}$ from the first term.

M1: Factorises out $\frac{4}{5}$ to form $= \frac{4}{5} \times \left(325\left(\frac{4}{5}\right)^{k-1} + 100k - 400\right) + ...$

A1: For correct work leading $U_{k+1} = 0.8U_k + 20(k+1)$

A1: For a completely correct proof (all previous marks must be gained) with a conclusion that includes all of the bold statements in the scheme or equivalents.
(c) **M1:** Evaluates $U_{24}\ U_{25}\ U_{26}$ or U_{27} . Allow attempts that deduce $\left(\frac{4}{5}\right)^n \to 0$ and just evaluate $100\times 26-400=2200$ **A1:** Correct value and appropriate conclusion.

Question	Scheme	Marks	AOs
4(a)	$168 = 66 \times 2 + 36$ $66 = 36 \times 1 + 30$ $36 = 30 \times 1 + 6$ $\{30 = 5 \times 6 + 0\}$	M1	2.1
	{Last non-zero remainder is 6} so highest common factor is 6.*	A1*	2.4
		(2)	
(b)	$6 = 36 - 30 \times 1$ $= 36 - (66 - 36 \times 1) = 2 \times 36 - 1 \times 66$ $= 2(168 - 66 \times 2) - 1 \times 66$	M1 A1	2.1 1.1b
	$\Rightarrow 6 = 168 \times 2 - 66 \times 5$ so $a = 2$, $b = -5$	A1	2.2a
		(3)	
(c)	As 168x + 66y will always be an integer multiple of 6, but **10 is not a multiple of 6**, there can be no integer solutions. or gcd(168, 66) = 6 and as $6 \nmid 10$	B1	2.4
		(1)	
(d)	From (b) we deduce $1 = 28 \times 2 - 11 \times 5$	B1	2.2a
	Hence $8 \times 11 \times -5 \equiv -5 \times 8 \pmod{28}$	M1	1.1b
	$v \equiv -40 \equiv 16 \pmod{28}$	A1	1.1b
		(3)	
	Alternative 1 Forms an expression of the form $8 = 28A + 11B$	M1	1.1b
	e.g. $8 = 5 \times 28 - 12 \times 11$ $8 = 16 \times 28 - 40 \times 11$	B1	2.2a
	e.g. $v \equiv -12 \pmod{28}$ $v \equiv -40 \pmod{28}$	A1	1.1b
		(3)	
	Alternative 2 $11v \equiv 8 \pmod{28} \Rightarrow 44v \equiv 32 \pmod{28} \Rightarrow 11v \equiv 8 \pmod{7}$	B1	1.1b
	$v \equiv 11^5 \times 8 \pmod{7} \Rightarrow \{v \equiv 2 \pmod{7}\}$	M1	1.1b
	$v \equiv 16 \pmod{28}$	A1	2.2a
		(3)	
			(9 marks)

Notes:

(a)

M1: Attempt at the Euclidean algorithm to find the gcd of 168 and 66, condone a numerical slip.

A1*: A fully correct application of the Euclidean algorithm and explains that the highest common factor is 6.

(b)

M1: Attempts to apply back substitution to find suitable values.

A1: Correct expression in just the numbers 168 and 66 reached (all remainders eliminated). Need not be fully simplified.

A1: Deduces the correct values of *a* and *b*. Allow if only seen in the expression.

(c)

B1: Correct explanation. Must refer to 10 not being a multiple of 6.

(d)

B1: Uses the result in (b) and divides through by 6 to form a correct statement between 11 and 28. Alternatively uses Euclidean algorithm for 28 and 11 to form an equation for 1 as multiples of 28 and 11. Allow this mark for a correct unsimplified equation.

M1: Multiplies through in their equation by 8 or the original equation through by -5 (or other multiplicative inverse).

A1: Deduces the correct solutions. Accept either $-40 \pmod{28}$ or 16 (mod 28), or any other correct congruence.

Alternative 1: note the order of marks has changed

M1: forms an equation for 8 in terms of multiples of 28 and 11

B1: Correct equation, allow this mark for a correct unsimplified equation.

A1: Correct solution

Alternative 2:

B1: Correct equation

M1: Uses Fermat's little theorem

A1: Correct solution

Trial and error approach

(d) – likely to score B0M1A1.

B1: Gives some kind of proof that the solution is unique. E.g. by reference to hcf(11,28) = 1 or trial of every possible value.

M1: Applies trial and improvement method to find one solution.

A1: Correct solution found.

Correct answer stated with no supportive working scores B0M1A1

Question	Scheme	Marks	AOs
5(i)	**(a)** $5^4 \times ...$ or $... \times 4 \times 3 \times 2$ or $... \times {}^4P_3$	M1	1.1b
	$5^4 \times 4 \times 3 \times 2 = 15000$	A1	1.1b
	(b) The structure is _N_N_N_N_, where three _'s are letters, so have ${}^5C_3$ choices for the letters, hence new number of combinations is ${}^5C_3 \times 15000$ or 10×15000	M1	3.1a
	So increase in number of codes is 135000	A1ft	1.1b
		(4)	
(ii)	**(a)** Divisible by $9 \Rightarrow a+b+c+d = 9k$ or e.g. attempt 1+3+5+7=16, not a multiple of 9, so reject. E.g. $1+3+5+7+9 = 25$ so k = 1 or 2, but $1+3+5+7 = 16$ >9, so k = 2 hence $\Rightarrow 25-(a+b+c+d) = 25-18 = 7$ so 7 missing. Or considers options in turn and finds only $1+3+5+9 = 18$ works	M1	3.1a
	So the digits are 1, 3, 5 and 9	A1	1.1b
	(b) Combination is either 1359 or 9531 and $1359 (\text{mod} 13) \equiv 59 (\text{mod} 13) \equiv 7$, $9531 (\text{mod} 95) \equiv 31$	M1	1.1b
	Combination is 1359	A1	2.2a
		(4)	
			(8 marks)

Notes:

(i)(a)

M1: $5^4 \times ...$ or $... \times 4 \times 3 \times 2$ or $... \times {}^4P_3$

A1: For 15000

(b)

M1: Correct strategy to find the new number of codes seen or implied.

A1ft: Increase in codes is 135000. Follow through $9 \times$ their 15000

(ii)(a)

M1: Full strategy to deduce the correct set of digits.

A1: Digits are 1,3, 5 and 9

Writes down the correct values scores M1A1

(b)

M1: Checks which of the two possible combinations satisfies the final property.

A1: Combination is 1359

Writes down the correct code scores M1A1

Question	Scheme	Marks	AOs
6	$\lambda^2 = 2\lambda - \frac{4}{3} \Rightarrow 3\lambda^2 - 6\lambda + 4 = 0 \Rightarrow \lambda = \ldots$	M1	1.1b
	$\lambda = \frac{6 \pm \sqrt{36-48}}{6} = \frac{3 \pm \mathrm{i}\sqrt{3}}{3} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} \pm \frac{1}{2}\mathrm{i}\right) = \frac{2\sqrt{3}}{3}\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{6}}$	A1	1.1b
	CF is $u_n = A\left(\frac{3+\mathrm{i}\sqrt{3}}{3}\right)^n + B\left(\frac{3-\mathrm{i}\sqrt{3}}{3}\right)^n$ o.e. or $\left(\frac{2\sqrt{3}}{3}\right)^n \left(P\cos\left(\frac{\pi n}{6}\right) + Q\sin\left(\frac{\pi n}{6}\right)\right)$	A1ft	2.2a
	$u_n = kn + l \Rightarrow k(n+2) + l = 2k(n+1) + 2l - \frac{4}{3}(kn+l) + n$ $\Rightarrow \left(1 - \frac{1}{3}k\right)n - \frac{1}{3}l = 0 \Rightarrow k = \ldots, l = \ldots \quad (k = 3, l = 0)$	M1	1.1b
	$u_n = A\left(\frac{3+\mathrm{i}\sqrt{3}}{3}\right)^n + B\left(\frac{3-\mathrm{i}\sqrt{3}}{3}\right)^n + 3n$ o.e. or $u_n = \left(\frac{2\sqrt{3}}{3}\right)^n \left(P\cos\left(\frac{\pi n}{6}\right) + Q\sin\left(\frac{\pi n}{6}\right)\right) + 3n$	A1	1.1b
	$\left.\begin{matrix} u_0 = 1 \Rightarrow A + B = 1 \\ u_1 = 4 \Rightarrow A + B + (A-B)\frac{\mathrm{i}\sqrt{3}}{3} = 1 \end{matrix}\right\} \Rightarrow (A-B)\frac{\mathrm{i}\sqrt{3}}{3} = 0 \Rightarrow A = \ldots, B = \ldots$ $\left.\begin{matrix} u_0 = 1 \Rightarrow P + 0Q = 1 \\ u_1 = 4 \Rightarrow \frac{2}{\sqrt{3}}\left(P\frac{\sqrt{3}}{2} + \frac{Q}{2}\right) = 1 \end{matrix}\right\} \Rightarrow P = \ldots \Rightarrow Q = \ldots$ Solves simultaneous equations to find values for A and B or P and Q	M1	3.1a
	$\left(A = B = \frac{1}{2} \text{ or } P = 1, Q = 0\right)$ $\Rightarrow u_n = \frac{1}{2}\left(\frac{3+\mathrm{i}\sqrt{3}}{3}\right)^n + \frac{1}{2}\left(\frac{3-\mathrm{i}\sqrt{3}}{3}\right)^n + 3n$ o.e. Or $u_n = \left(\frac{2\sqrt{3}}{3}\right)^n \cos\left(\frac{\pi n}{6}\right) + 3n$	A1	1.1b
		(7)	
			(7 marks)
Notes:			

M1: Forms and solves the auxiliary equation.

A1: Correct roots – either Cartesian or polar form, award when first seen and isw.

A1ft: Correct complementary function, follow through on their first complex roots. (so A1A0 if roots initially correct but error simplifying leads to wrong CF). Note: use of power $n + 1$ or $n - 1$ in Cartesian form is also fine

M1: Correct form for the particular solution and a complete method to find the PS.

A1: Correct general solution, either form

M1. Substitutes $n = 1$ and sets equal to 4 **and** substitutes $n = 0$ and sets equal to 1. To find the values of the constants

A1: Correct solution, any form.

Question	Scheme	Marks	AOs
7 (a)	$x \bullet e = x \Rightarrow 3(x+e+1)+2xe = x$ $\Rightarrow 3e+2xe = x-3x-3 \Rightarrow e = ...$ Or Let $e=-1 \Rightarrow e \bullet y = 3(-1+y+1)+2(-1)y = ...$	M1	2.1
	$\Rightarrow e = -\frac{2x+3}{2x+3} \Rightarrow e=-1$ Or $e \bullet y = 3(-1+y+1)+2(-1)y = y$ therefore $e=-1$	A1	2.2a
		(2)	
(b)	Let the inverse of x be y then $x \bullet y = e \Rightarrow 3(x+y+1)+2xy = -1$	M1	3.1a
	$\Rightarrow 3y+2xy = -1-3x-3 \Rightarrow y = ...$	M1	2.1
	$\Rightarrow x^{-1} = -\frac{3x+4}{2x+3}$ oe such as $-\frac{3}{2}+\frac{1}{4x+6}$	A1	2.2a
		(3)	
(c)	E.g. for $x=-\frac{3}{2}$ the identity property would fail since we cannot divide by $2x+3=0$ or shows $x^{-1} = -\frac{-\frac{1}{2}}{0}$ so no inverse or can't divide by 0 Or $-\frac{3}{2} \bullet x = 3\left(-\frac{3}{2}+x+1\right)-3x = -\frac{3}{2}$ so the identity property fails as not unique as any x would be an identity for $-\frac{3}{2}$.	B1	2.4
		(1)	
		(6 Marks)	

Notes:

(a)
M1: Sets up and solves a correct equation for the identity element.
Alternatively spots the identity is -1 and attempts to show this satisfies the identity property.
E.g. $-1 \bullet x = 3(-1+x+1)+2(-1)x = ...$

A1: Correct identity / correct demonstration that -1 is the identity with appropriate conclusion.
(Note only one side needs to be checked as we are told G is a group so one side is sufficient by uniqueness).

Special case: states $e = -1$ with no working scores M1A0
(b) **M1:** Sets up a correct equation for the inverse using their identity element. May use x^{-1} throughout, or may use y or other variable. **M1:** Expands, rearranges and by factorising out the inverse element from two terms to find an expression for the inverse element. **A1:** Correct inverse element. Accept equivalent forms.
(c) **B1:** Explains that the identity property would fail for $x = -\frac{3}{2}$. Alternatively accept a similar argument that $x = -\frac{3}{2}$ could not have an inverse due to dividing by zero. May refer back to work in (b), if done there, without need to repeat it. Just saying undefined is B0 without a reason why. **Note:** Reasons saying closure or associativity would fail are incorrect so B0, as both of these properties are satisfied on all of $\mathbb{R}$

Question	Scheme	Marks	AOs
8 (a)	$I_n = \int_0^2 (x-2)^n e^{4x}\,dx = \left[(x-2)^n \times Ae^{4x}\right]_0^2 - \int_0^2 n(x-2)^{n-1} \times Ae^{4x}\,dx$ $\left\{= \left[(x-2)^n \times \frac{1}{4}e^{4x}\right]_0^2 - \int_0^2 n(x-2)^{n-1} \times \frac{1}{4}e^{4x}\,dx\right\}$	M1	1.1b
	$= \left(0 - A(-2)^n\right) - A\int_0^2 (x-2)^{n-1}e^{4x}\,dx$ $\left\{= \left(0 - \frac{1}{4}(-2)^n\right) - \frac{n}{4}\int_0^2 (x-2)^{n-1}e^{4x}\,dx\right\}$	M1	1.1b
	$= -\frac{1}{(-2)^2}(-2)^n - \frac{n}{4}I_{n-1}$	M1	3.1a
	$I_n = -(-2)^{n-2} - \frac{n}{4}I_{n-1}$	A1	2.1
		(4)	
	Alternative: $I_n = \int_0^2 (x-2)^n e^{4x}\,dx = \left[\frac{(x-2)^{n+1}}{n+1} \times e^{4x}\right]_0^2 - \int_0^2 \frac{(x-2)^{n+1}}{n+1} \times Ae^{4x}\,dx$ $\left\{= \left[\frac{(x-2)^{n+1}}{n+1} \times e^{4x}\right]_0^2 - \int_0^2 \frac{(x-2)^{n+1}}{n+1} \times 4e^{4x}\,dx\right\}$	M1	1.1b
	$= \left(0 - \frac{(-2)^{n+1}}{n+1}\right) - \frac{A}{n+1}\int_0^2 (x-2)^{n+1}e^{4x}dx$ $\left\{= \left(0 - \frac{(-2)^{n+1}}{n+1}\right) - \frac{4}{n+1}\int_0^2 (x-2)^{n+1}e^{4x}dx\right\}$	M1	1.1b
	$\Rightarrow (n+1)I_n = -(-2)^{n+1} - 4I_{n+1} \Rightarrow I_{n+1} = -\frac{1}{(-2)^2}(-2)^{n+1} - \frac{n+1}{4}I_n$	M1	3.1a
	$\Rightarrow I_n = -(-2)^{n-2} - \frac{n}{4}I_{n-1}$	A1	2.1
		(4)	
(b)	$I_2 = \int_0^2 (x-2)^2 e^{4x}dx = -(-2)^0 - \frac{2}{4}I_1 = -1 - \frac{1}{2}\left(-(-2)^{-1} - \frac{1}{4}I_0\right)$	M1	2.1
	$= -1 - \frac{1}{4} + \frac{1}{8}\left(\frac{1}{4}e^8 - \frac{1}{4}\right)$	M1	1.1b
	$= \frac{1}{32}e^8 - \frac{41}{32}$	A1	1.1b
		(3)	
			(7 marks)
Notes:			

(a)

M1: Applies integration by parts to achieve the correct form

M1: Substitutes in the limits 0 and 2 and simplifies integral to match the form of I_n

M1: Writes 4 as $(-2)^2$ in the first terms (or correctly combines powers) and replaces integral by I_{n-1}

In the alternative award this M for a full process to rearrange to the form I_{n+1} as well as the above constraints.

A1: Achieves the correct answer following correct working, cso. In the alternative a replacement of *n* by *n* – 1 must also occur. isw

(b)

M1: For a full process of reducing the integral to an expression in I_0

M1: Evaluates I_0 and substitutes into the expression.

A1: cso $\frac{1}{32}e^8 - \frac{41}{32}$ or simplified equivalent.

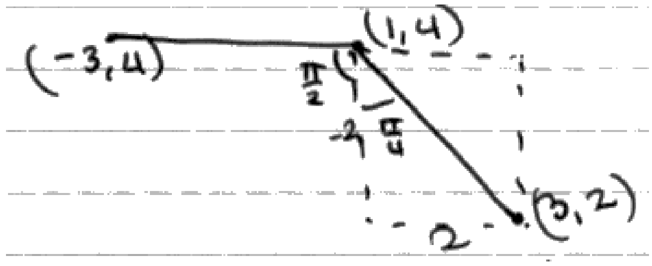
Note: Candidates might use

M1: $I_2 = \int_0^2 (x-2)^2 e^{4x}\,dx = -(-2)^0 - \frac{2}{4}I_1 = -1 - \frac{1}{2}I_1$

M1: $I_1 = \int_0^2 (x-2)e^{4x}\,dx = -\frac{1}{16}e^8 + \frac{9}{16}$ therefore $I_2 = -1 - \frac{1}{2}\left(-\frac{1}{16}e^8 + \frac{9}{16}\right)$

A1: cso $\frac{1}{32}e^8 - \frac{41}{32}$ or simplified equivalent. isw

Question	Scheme	Marks	AOs
9(a)	Centre lies on perpendicular bisector of z_2 and z_3, which is vertical as same imaginary components. Hence $\frac{a+1}{2}=-1 \Rightarrow a=...$ Or $(a-(-1))^2+(4-b)^2=(1-(-1))^2+(4-b)^2 \Rightarrow a=...$ $(a-(-1))^2=(1-(-1))^2 \Rightarrow a=...$ Or $-1-a=1-(-1) \Rightarrow a=...$	M1	1.1b
	$a=-3$	A1	2.2a
		(2)	
	Alternative $(x-(-1))^2+(y-b)^2=r^2$ $\left.\begin{array}{l}(1+1)^2+(4-b)^2=r^2 \\ (3+1)^2+(2-b)^2=r^2\end{array}\right\} \Rightarrow 4+(4-b)^2=16+(2-b)^2 \Rightarrow b=...\{0\}$ $r^2=4+(4-"0")^2=...\{20\}$ $(a+1)^2+(4-"0")^2="20" \Rightarrow a=...$	M1	1.1b
	$a=-3$ only the other root must be rejected	A1	2.2a
		(2)	
(b)	Equation has form $\arg\left(\frac{z-z_1}{z-z_2}\right)=\theta$	B1	1.2
	$\arg\left(\frac{1+4\text{i}-(3+2\text{i})}{1+4\text{i}-("-3"+4\text{i})}\right)=\arg(-2+2\text{i})-\arg(4)=...\left(=\frac{3\pi}{4}\right)$ Or $\arg\left(\frac{1+4\text{i}-3-2\text{i}}{1+4\text{i}-a-4\text{i}}\right)=\arg\left(\frac{-2+2\text{i}}{4}\right)=\arg\left(-\frac{1}{2}+\frac{1}{2}\text{i}\right)=\frac{3\pi}{4}$ or $\overrightarrow{z_3z_2}=\begin{pmatrix}-4\\0\end{pmatrix}$ and $\overrightarrow{z_3z_1}=\begin{pmatrix}2\\-2\end{pmatrix}$ $\cos\theta=\frac{\begin{pmatrix}-4\\0\end{pmatrix}\bullet\begin{pmatrix}2\\-2\end{pmatrix}}{4\sqrt{2^2+(-2)^2}} \Rightarrow \theta=...$ Or	M1	3.1a

	$\cos\theta = \dfrac{4^2+8-40}{2\times4\times\sqrt{8}} \Rightarrow \theta = ...$ or Using trigonometry 		
	Equation is $\arg\left(\dfrac{z-3-2\text{i}}{z+3-4\text{i}}\right) = \dfrac{3\pi}{4}$ o.e.e. Equation is $\arg(z-3-2\text{i}) - \arg(z+3-4\text{i}) = \dfrac{3\pi}{4}$ o.e.e.	A1	2.1
		(3)	
(5 marks)			

Notes:

(a)
M: Forms a correct strategy to find the value of *a*, see scheme for various approaches
A1: Correct value

(b)
B1: Recalls the correct form for the equation of an arc, with any angle or θ. Look for the correct form, so allow if z_1 and z_2 are the other way round.

M1: Any correct full method to find the value of θ, see scheme for various approaches
A1: Correct equation.
Correct answer implies full marks

Question	Scheme	Marks	AOs
10(a)	$y=\sqrt{1+\frac{x^2}{9}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(1+\frac{x^2}{9}\right)^{-\frac{1}{2}}\times\frac{2x}{9}=\frac{x}{9}\left(1+\frac{x^2}{9}\right)^{-\frac{1}{2}}$	M1 A1	1.1b 1.1b
	Surface of revolution $S=2\pi\int y\sqrt{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)}\,\mathrm{d}x=2\pi\int\sqrt{1+\frac{x^2}{9}}\sqrt{\left(1+\frac{x^2}{81}\left(1+\frac{x^2}{9}\right)^{-1}\right)}\mathrm{d}x$	M1	2.1
	For example $=2\pi\int\frac{1}{3}\sqrt{9+x^2}\sqrt{\left(\frac{9(9+x^2)+x^2}{9(9+x^2)}\right)}\mathrm{d}x=2\pi\int\frac{1}{3}\sqrt{9+x^2}\sqrt{\left(\frac{81+10x^2}{9(9+x^2)}\right)}\mathrm{d}x$ Or $=2\pi\int\sqrt{\left(1+\frac{x^2}{9}+\frac{x^2}{81}\right)}\,\mathrm{d}x=\frac{2\pi}{9}\int\sqrt{(81+9x^2+x^2)}\,\mathrm{d}x$	M1	1.1b
	Circular end has area $\pi\times y^2=\pi\left(1+\frac{16}{9}\right)=\frac{25\pi}{9}$	B1	2.2a
	So $S=\frac{2\pi}{9}\int_{-4}^{4}\sqrt{81+10x^2}\,\mathrm{d}x+\frac{50\pi}{9}$	A1	3.4
		(6)	
	Alternative $y^2=1+\frac{x^2}{9}\Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{9}x$	M1 A1	
	Surface of revolution $S=2\pi\int y\sqrt{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)}\,\mathrm{d}x=2\pi\int\sqrt{\left(y^2+\left(y\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)}\,\mathrm{d}x$ $=2\pi\int\sqrt{\left(1+\frac{x^2}{9}+\left(\frac{x}{9}\right)^2\right)}\,\mathrm{d}x$	M1	
	$=2\pi\int\sqrt{\left(1+\frac{x^2}{9}+\frac{x^2}{81}\right)}\,\mathrm{d}x=\frac{2\pi}{9}\int\sqrt{(81+9x^2+x^2)}\,\mathrm{d}x$	M1	
	Circular end has area $\pi\times y^2=\pi\left(1+\frac{16}{9}\right)=\frac{25\pi}{9}$	B1	
	So $S=\frac{2\pi}{9}\int_{-4}^{4}\sqrt{81+10x^2}\,\mathrm{d}x+\frac{50\pi}{9}$	A1	
		(6)	

(b)	$x=\frac{9}{\sqrt{10}}\sinh u \Rightarrow \frac{dx}{du}=\frac{9}{\sqrt{10}}\cosh u$	B1	1.1b
	So $S="\frac{2}{9}"\pi\int\sqrt{81+81\sinh^2 u}\,\frac{9}{\sqrt{10}}\cosh u\,du$	M1	2.1
	$=B\pi\int\cosh^2 u\,du=B\pi\int\frac{1}{2}(\pm1\pm\cosh 2u)\,du$	M1	1.1b
	$="\frac{2}{9}"\times\frac{81\pi}{\sqrt{10}}\left[\frac{u}{2}+\frac{1}{4}\sinh 2u\right]$	A1ft	1.1b
	So $S=\frac{50\pi}{9}+"\frac{2}{9}"\times\frac{81\pi}{\sqrt{10}}\left[\begin{array}{l}\frac{1}{2}\text{arsinh}\left(\frac{4\sqrt{10}}{9}\right)+\frac{1}{4}\sinh 2\text{arsinh}\left(\frac{4\sqrt{10}}{9}\right)\\-\left(\frac{1}{2}\text{arsinh}\left(\frac{-4\sqrt{10}}{9}\right)+\frac{1}{4}\sinh 2\text{arsinh}\left(\frac{-4\sqrt{10}}{9}\right)\right)\end{array}\right]=\ldots$ $S=\frac{50\pi}{9}+"\frac{2}{9}"\times\frac{81\pi}{\sqrt{10}}\left[(1.7827\ldots)-(-1.7827\ldots)\right]=\ldots$	M1	3.4
	Surface area is awrt 81 (cm^2)	A1	1.1b
		(6)	
(12 marks)			

Notes:

(a)

M1: Attempts to find $\frac{dy}{dx}$ achieving the form $Ax\left(1+\frac{x^2}{9}\right)^{-\frac{1}{2}}$ oe

A1: Correct derivative.

M1: Uses the formula surface area $=2\pi\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

M1: Manipulates the integral and simplifies to the given form.
B1: Correct area for a circular end found.
A1: Achieves the correct answer with no errors seen, including the limits from the model.
Alternative

M1: Find y^2 and differentiates to the form $2y\frac{dy}{dx}=Ax$

A1: Correct derivative.

M1: Uses the formula surface area $S=2\pi\int y\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}\,dx=2\pi\int\sqrt{\left(y^2+\left(y\frac{dy}{dx}\right)^2\right)}\,dx$

M1: Manipulates the integral and simplifies to the given form.
B1: Correct area for a circular end found.

A1: Achieves the correct answer with no errors seen, including the limits from the model.
(b) **B1:** Correct derivative statement connecting x and u **M1:** Makes a full substitution to obtain an integral in terms of u only. No need for limits for this mark, and may use p or their p from (a) **M1:** Simplifies and applies double angle formula of the form $\cosh 2u = \pm 1 \pm \cosh^2 u$ to achieve an integral of the form $B\pi\int \frac{1}{2}(\pm 1 \pm \cosh 2u)\,du$ **A1ft:** For correct integration with their p from (a). **M1:** Applies appropriate limits for their integral to find the surface area and adds the area of the ends. Either −4 and 4 if returning to integral in terms of x (or 0 and 4 and doubling) or $\text{arsinh}\left(\frac{\pm 4\sqrt{10}}{9}\right)$ (or decimal approximation 1.14) if using u. May be implied by a correct answer if explicit substitution not seen. FYI the integral (without ends added) evaluates to 63.758… A**1:** Correct surface area.

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