Integration - Integration by substitution, by parts and by partial fractions

Name: ______________________________

Class: ______________________________

Date: ______________________________

Time:

Total marks available:

Total marks achieved: ______

A Level Mathematics : Pure Mathematics

Subject: Mathematics

Topic 8 : Integration - Integration by substitution, by parts and by partial fractions

Type: Topic Questions

To be used by all students preparing for Edexcel A Level Mathematics - Students of other Boards may also find this useful

Q1.

(a) Express $\frac{3}{(2x-1)(x+1)}$ in partial fractions.

(3)

When chemical *A* and chemical *B* are mixed, oxygen is produced.

A scientist mixed these two chemicals and measured the total volume of oxygen produced over a period of time.

The total volume of oxygen produced, V m^3, t hours after the chemicals were mixed, is modelled by the differential equation

$$\frac{dV}{dt} = \frac{3V}{(2t-1)(t+1)} \qquad V \geqslant 0 \quad t \geqslant k$$

where k is a constant.

Given that exactly 2 hours after the chemicals were mixed, a total volume of 3 m^3 of oxygen had been produced,

(b) solve the differential equation to show that

$$V = \frac{3(2t-1)}{(t+1)}$$

(5)

The scientist noticed that

- there was a **time delay** between the chemicals being mixed and oxygen being produced
- there was a **limit** to the total volume of oxygen produced

Deduce from the model

(c) (i) the **time delay** giving your answer in minutes,

(ii) the **limit** giving your answer in m^3

(2)

(Total for question = 10 marks)

Q2.

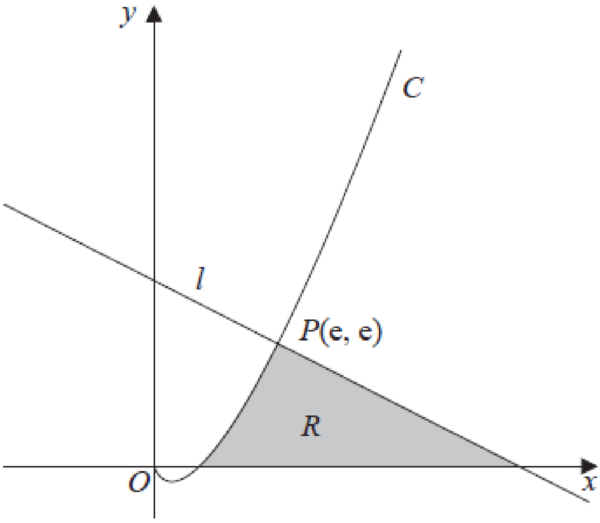

Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x \ln x, \quad x > 0$

The line l is the normal to C at the point $P(\text{e}, \text{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\text{e}^2 + B$ where A and B are rational numbers to be found.

(10)

(Total for question = 10 marks)

Q3.

The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

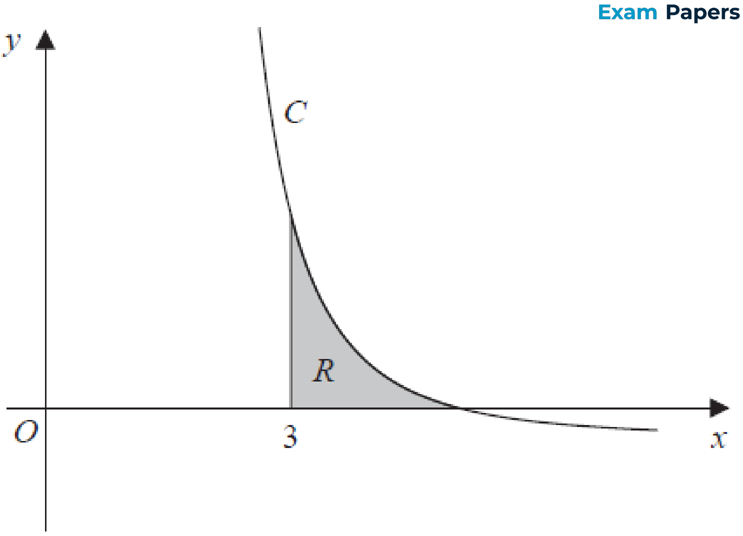

Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a\ln 2 + b\ln 3$, where a and b are rational constants to be found.

(8)

(Total for question = 11 marks)

Q4.

(a) Express $\dfrac{1}{P(11-2P)}$ in partial fractions.

(3)

A population of meerkats is being studied.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{22}P(11-2P), \quad t \geqslant 0, \qquad 0 < P < 5.5$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that there were 1000 meerkats in the population when the study began,

(b) determine the time taken, in years, for this population of meerkats to double,

(6)

(c) show that

$$P = \frac{A}{B + Ce^{-\frac{1}{2}t}}$$

where *A*, *B* and *C* are integers to be found.

(3)

(Total for question = 12 marks)

Q5.

(a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_p^q \frac{2(u-1)^3}{u}\,du$$

where *p* and *q* are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = A - B\ln 5$$

where *A* and *B* are constants to be found.

(4)

(Total for question = 7 marks)

Q6.

(a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{dh}{4-\sqrt{h}} = -8\ln\left|4 - \sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

(Total for question = 15 marks)

Q7.

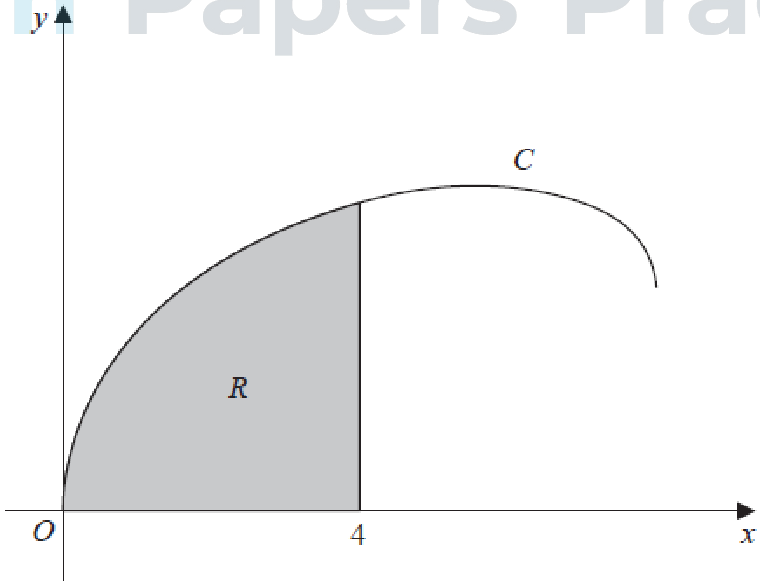

Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \le t \le \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a (8 - 8\cos 4t + 48\sin^2 t\cos t)\,dt$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

(Total for question = 9 marks)

Q8.

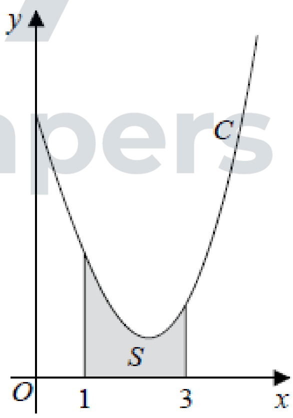

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the line with equation $x = 1$, the x-axis and the line with equation $x = 3$

The table below shows corresponding values of x and y with the values of y given to 4 decimal places as appropriate.

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of S, giving your answer to 3 decimal places.

(3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S.

(1)

(c) Show that the exact area of S can be written in the form $\frac{a}{b} + \ln c$, where a, b and c are integers to be found.

(6)

(*In part c, solutions based entirely on graphical or numerical methods are not acceptable.*)

(Total for question = 10 marks)

Q9.

Show that

$$\int_0^2 2x\sqrt{x+2}\,dx = \frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

(Total for question = 7 marks)