IB Maths: AA HL

6.2 Extended Questions (Section B, HL)

Topic Questions

These practice questions can be used by students and teachers and is Suitable for IB Maths AA HL Topic Questions

Course	IB Maths
Section	6. Extended Questions
Topic	6.2 Extended Questions (Section B, HL)
Difficulty	Medium

Level: IB Maths

Subject: IB Maths AA HL

Board: IB Maths

Topic: 6.2 Extended Questions (Section B, HL)

Question 1

The function f is defined by $f(x) = \frac{2x-1}{x^2+3x-4}$, for $x \in \mathbb{R}$, $x \neq m$, $x \neq n$.

a)
Find the values of m and n.

[2 marks]

b)
Find an expression for $f'(x)$.

[3 marks]

The graph of $y = f(x)$ has exactly one point of inflection.

c)
Find the x-coordinate of the point of inflection.

[2 marks]

d)
Sketch the graph of $y = f(x)$ for $-6 \leq x \leq 6$, showing the coordinates of any axis intercepts and local maxima and local minima, and giving the equations of any asymptotes.

[4 marks]

The function g is defined by $g(x) = \frac{x^2+3x-4}{2x-1}$, for $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

e)
Find the equation of the oblique asymptote of the graph of $y = g(x)$.

[3 marks]

f)

By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $g(x) < f(x)$ for $x \in \mathbb{R}$.

[4 marks]

Question 2

The function f has a derivative given by $f'(x) = \dfrac{1}{3x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, where k is a positive constant.

a)

The expression for $f'(x)$ can be written in the form $\dfrac{a}{3x} + \dfrac{b}{k-x}$ where $p, q \in \mathbb{R}$. Find a and b in terms of k.

[3 marks]

b)

Hence find an expression for $f(x)$.

[3 marks]

R is the population of rabbits on an island. The rate of change of the population can be modelled by the differential equation $\dfrac{\mathrm{d}R}{\mathrm{d}t} = \dfrac{3R(k-R)}{4k}$, where t is the time measured in years, $t \geq 0$, and k is the maximum population that the island can support.

The initial population of the rabbits is 20.

c)

By solving the differential equation, show that $R = \dfrac{20ke^{\frac{3}{4}t}}{k - 20 + 20e^{\frac{3}{4}t}}$

[7 marks]

After two years, the population of rabbits has risen to 70.

d)
Find k.

[3 marks]

e)
Find the value of t at which the population of rabbits is growing at its fastest rate.

[2 marks]

Question 3

A particle is moving in a vertical line and its acceleration, in ms^{-2} , at time t seconds, $t \geq 0$ is given by $a = -\frac{1-v}{2}$, where v is the velocity in meters per second and $v < 1$.

The particle starts at a fixed origin O with initial velocity v_o ms^{-1}.

a)
By solving a suitable differential equation, show that the particle's velocity at time t is given by $v(t) = 1 - e^{-\frac{t}{2}}(1 - v_o)$.

[6 marks]

The particle moves down in the negative direction, until its displacement relative to the origin reaches a minimum. Then the particle changes direction and starts moving up, in a positive direction.

b)
(i)
If the initial velocity of the particle is -3 ms^{-1}, find the time at which the minimum displacement of the particle from the origin occurs, giving your answer in exact form.

(ii)
If T is the time in seconds when the displacement reaches its smallest value, show that $T = 2\ln(1 - v_o)$.

[4 marks]

c)

(i)

Find a general expression for the displacement, in terms of t and v_o.

(ii)

Combine this general expression with the result from part (b)(ii) to find an expression for the minimum displacement of the particle in terms of v_o.

[5 marks]

Let $v(T-k)$ represent the particle's velocity k seconds before the minimum displacement and $v(T+k)$ the particle's velocity k seconds after the minimum displacement.

d)

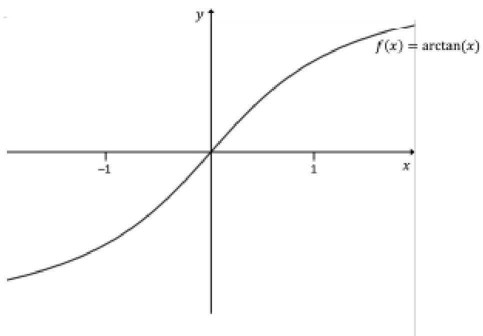
(i) Show that $v(T-k) = 1 - e^{\frac{k}{2}}$.

(ii) Given that $v(T+k) = 1 - e^{-\frac{k}{2}}$, show that $v(T-k) + v(T+k) \geq 0$.

[5 marks]

Question 4

The diagram below shows the graph of $f(x) = \arctan(x)$, $x \in \mathbb{R}$. The graph has rotational symmetry of order 2 about the origin.



a)

A different function, g, is described by $g(x) = -\arctan(x-1)$, $x \in \mathbb{R}$.

(i)

Describe the sequence of transformations that transforms $f(x)$ to $g(x)$.

(ii)

Sketch the graph of $g(x)$ on the axes above.

(iii)

Using your answers to parts (i) and (ii) to help you, describe the relationship between $\int_0^1 \arctan(x)\,dx$ and $\int_0^1 -\arctan(x-1)\,dx$. .

[5 marks]

b)

(i)

Prove that $\arctan p - \arctan q = \arctan\left(\dfrac{p-q}{1+pq}\right)$.

(ii) Show that $\arctan\left(\dfrac{1}{x^2-x+1}\right)$ can be written as $\arctan(x) - \arctan(x-1)$.

[6 marks]

c)

Using the results from parts (a) and (b), evaluate $\int_0^1 \arctan\left(\dfrac{1}{x^2-x+1}\right)dx$, leaving your answer in exact form.

[7 marks]

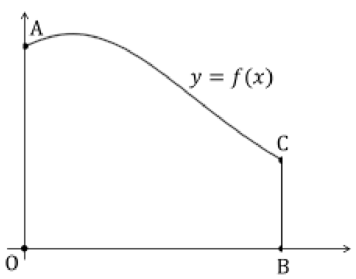
Question 5

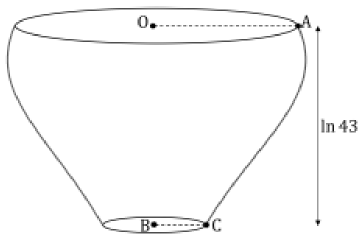
Paola is modelling a small vase from her house for her maths project. To model the edge of the vase in cross-section, she decides to use a function f of the form

$$f(x) = \frac{qe^{\frac{x}{2}}}{2+e^x}$$

where $x \in \mathbb{R}$, $x \geq 0$ and $q \in \mathbb{R}^+$.

The function and the vase are represented in the diagrams below.





The vertical height of the vase, OB, is measured along the x-axis. The radius of the vase's opening is OA, and its base radius is BC.

To model the vase, she will rotate by 2π radians about the x-axis the region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis, and the line $x = \ln 43$.

a)

Show that the volume of the solid of revolution thus formed is $\frac{14q^2\pi}{45}$ units^3.

[6 marks]

The volume of the actual vase is 100 cm^3.

b)

Use this information to find the value of q.

[2 marks]

c)

Find the cross-sectional radius of the vase

(i) at its base,

(ii) at its widest point.

[4 marks]

Paola wants to investigate how the cross-sectional radius of the vase changes.

d)

Sketch a graph of the derivative of f, and use it to find the value of x at which the cross-sectional radius of the vase is decreasing most rapidly.

[4 marks]