6.1 Extended Questions

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Course	DP IB Maths
Section	6. Extended Questions
Topic	6.1 Extended Questions
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1

a) The median can be found using the cumulative frequency (c.f.) graph.

Median c.f. $= \frac{\text{Total}}{2}$

Median c.f. $= \frac{120}{2}$

Median c.f. $= 60$

Horizontal line at c.f. $= 60$

$\boxed{\therefore \text{Median} = \$52}$

b) Interquartile range

$IQR = Q_3 - Q_1$ (in formula booklet)

Horizontal lines at Q_3 and Q_1 c.f.

Q_3 c.f. $= \frac{3}{4} \times 120$

Q_3 c.f. $= 90$ $\therefore Q_3 = 64$

Q_1 c.f. $= \frac{1}{4} \times 120$

Q_1 c.f. $= 30$ $\therefore Q_1 = 36$

$IQR = 64 - 36$

$\boxed{IQR = \$28}$

c) $\frac{2}{3}$ of customers' baskets > \$p

$\therefore$ $\frac{1}{3}$ of customers' baskets < \$p

$c.f. = 120 - \left(\frac{2}{3} \times 120\right)$

$c.f. = 40$

Horizontal line at c.f. = 40.

$\boxed{\therefore p = \$42}$

d) Find q and r by drawing vertical lines using the given intervals.

$q = 36 - 9$

$\boxed{q = 27 \text{ customers}}$

$r = 100 - 36$

$\boxed{r = 64 \text{ customers}}$

e) \$50 corresponds to cf = 56.

$120 - 56 = 64$

$\therefore \frac{64}{120}$ spent over \$50.

$\text{estimate} = \frac{64}{120} \times 3600$

$\text{estimate} = 64 \times \frac{\cancel{3600}^{30}}{\cancel{120}}$

$\text{estimate} = 64 \times 30$

$\boxed{\text{estimate} = 1920 \text{ customers}}$

f) i) Only sampling customers at lunchtimes on weekdays is unlikely to provide a good representation of all customers.

ii) There are many possible answers.

A form of quota sampling, with a set number of customers randomly chosen throughout all opening hours on all days of the week.

Question 2

a) y-intercepts occur when $x = 0$

$f(0) = \frac{1}{3}(0)^3 - 4(0)^2 + 9(0) + 12$

$f(0) = 12$

$\boxed{A(0, 12)}$

b) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$f'(x) = \cancel{(3)} \cdot \frac{1}{\cancel{3}} x^2 - (2) \cdot 4x + 9$

$\boxed{f'(x) = x^2 - 8x + 9}$

c) $f'(x) = 0$ when the graph of f has a local maximum or a local minimum.

$\boxed{\text{The graph of } f \text{ shows a local maximum and a local minimum. } \therefore \text{ There are two distinct real solutions.}}$

Alternatively

$f'(x)$ is a quadratic and when $\Delta > 0$ the equation $f'(x) = 0$ has two solutions.

Discriminant

$\Delta = b^2 - 4ac$ (in formula booklet)

$\Delta = (8)^2 - 4(1)(9) > 0$

$\boxed{\Delta = 28 > 0 \;\therefore \text{ two distinct real solutions.}}$

d) Sub $x = \frac{8-\sqrt{26}}{2}$ into $f'(x)$.

$$f'\left(\frac{8-\sqrt{26}}{2}\right) = \left(\frac{8-\sqrt{26}}{2}\right)^2 - {}^{4}\cancel{8}\left(\frac{8-\sqrt{26}}{\cancel{2}}\right) + 9$$

$$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45-8\sqrt{26}}{2} - 4\left(8-\sqrt{26}\right) + 9$$

$$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45}{2} - \frac{{}^{4}\cancel{8}\sqrt{26}}{\cancel{2}} - 4\left(8-\sqrt{26}\right) + 9$$

$$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45}{2} \cancel{-4\sqrt{26}} - 32 \cancel{+4\sqrt{26}} + 9$$

$$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45}{2} - 32 + 9$$

$$f'\left(\frac{8-\sqrt{26}}{2}\right) = -\frac{1}{2}$$

$\boxed{\therefore \text{ gradient at point B is } -\tfrac{1}{2}.}$

e) Perpendicular gradients

$$m_{\perp} = -\frac{1}{m}$$

$$m_{\perp} = -\frac{1}{\left(-\frac{1}{2}\right)} \qquad \therefore m_{\perp} = 2$$

Set $f'(x) = 2$ and find x.

$$2 = x^2 - 8x + 9$$

$$x^2 - 8x + 7 = 0$$

$$(x - 7)(x - 1) = 0$$

$\therefore$ x-coordinates of points C and D are $x = 1$ and $x = 7$.

f) x - coordinate of C is $x = 1$ or $x = 7$

and is in the interval $0 < x < \frac{8-\sqrt{26}}{2}$.

$\therefore$ x - coordinate of C is $x = 1$.

Find $f(1)$

$$f(1) = \frac{1}{3}(1)^3 - 4(1)^2 + 9(1) + 12$$

$$f(1) = \frac{52}{3} \qquad \therefore C\left(1, \frac{52}{3}\right)$$

Sub $C\left(1, \frac{52}{3}\right)$ and $m = 2$ into $y - y_1 = m(x - x_1)$.

$$y - \frac{52}{3} = 2(x - 1)$$

$$y = 2x - 2 + \frac{52}{3}$$

$$\boxed{y = 2x + \frac{46}{3}}$$

Question 3

a) $t = 0$ at beginning of the study.

Find $P(0)$.

$$P(0) = \frac{3000}{1 + 99\underbrace{e^{-k(0)}}_{=1}}$$

$$P(0) = \frac{3000}{1 + 99}$$

$$\boxed{P(0) = 30}$$

b) i) Because $k > 0$.

$$e^{-kt} = \frac{1}{e^{kt}} \qquad (t \to \infty,\ e^{-kt} \to 0)$$

As t becomes large, e^{-kt} tends towards zero.

ii) $$\lim_{t \to \infty} P(t) = \frac{3000}{1 + 99(0)}$$

$$\lim_{t \to \infty} P(t) = \frac{3000}{1}$$

Hence the max population predicted by the model is 3000.

c) Method 1: Chain rule

$y = g(u)$, where $u = f(x)$

$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}$ (in formula booklet)

$P(t) = \frac{3000}{1+99e^{-kt}} = 3000(1+99e^{-kt})^{-1}$

$P'(t) = -3000(1+99e^{-kt})^{-2}(-99ke^{-kt})$

Method 2: Quotient rule

$y = \frac{u}{v} \longrightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (in formula booklet)

$u = 3000 \qquad v = 1 + 99e^{-kt}$

$P'(t) = \frac{\cancel{(1+99e^{-kt})(0)} - (3000)(-99ke^{-kt})}{(1+99e^{-kt})^2}$

$$\boxed{P'(t) = \frac{3000 \times 99ke^{-kt}}{(1+99e^{-kt})^2}}$$

d) i) $e^{x} > 0$ for all (real) values of x.

$$\therefore P'(t) = \frac{3000 \times 99ke^{-kt}}{(1+99e^{-kt})^2} = \frac{\text{positive number}}{\text{positive number}} > 0$$

$P'(t) > 0$ for all values of t,
$\therefore P(t)$ is an increasing function.

ii) The graph of P has a horizontal asymptote at $y = 3000$.
Hence $P(t)$ is always increasing but never exceeds 3000.

e) i) $P(0) = 30 \quad \therefore P(2) = 60$

$$60 = \frac{3000}{1+99e^{-k(2)}}$$

$$1 + 99e^{-2k} = 50$$

$$99e^{-2k} = 49$$

$$e^{2k} = \frac{99}{49}$$

$$2k = \ln \frac{99}{49}$$

$$\boxed{k = \frac{1}{2}\ln\left(\frac{99}{49}\right)}$$

ii) Find $P'(0)$.

$$P'(0) = \frac{3000 \times 99ke^{-k(0)}}{(1+99e^{-k(0)})^2}$$

$$P'(0) = \frac{\cancel{3000}^{3} \times 99k}{_{10}\cancel{(100)^2}}$$

$$P'(0) = \frac{3}{10} \times 99k$$

Sub in $k = \frac{1}{2}\ln\left(\frac{99}{49}\right)$

$$P'(0) = \frac{3}{10} \times 99\left(\frac{1}{2}\ln\left(\frac{99}{49}\right)\right)$$

$$P'(0) = \frac{297}{10}\left(\frac{1}{2}\ln\left(\frac{99}{49}\right)\right)$$

$$P'(0) = 29.7 \times \frac{1}{2}\ln\left(\frac{99}{49}\right)$$

$$\boxed{14.85\ln\left(\frac{99}{49}\right) \text{ or } 29.7\ln\frac{\sqrt{99}}{7} \text{ rabbits/month}}$$

Question 4

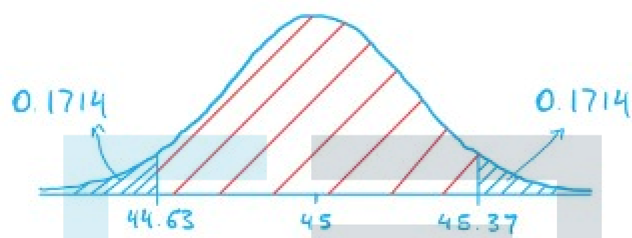
a) $X \sim N(45, \sigma^2)$

$45.37 - 45 = 0.37$

$45 - 44.63 = 0.37$

45.37 and 44.63 are equidistant from 45.

$P(X > 45.37) = P(X < 44.63) = 0.1714$



$P(44.63 < X < 45.37) = 1 - 2(0.1714)$

$P(44.63 < X < 45.37) = 0.6572$

$$\boxed{P(44.63 < X < 45.37) = 0.657 \text{ (3sf)}}$$

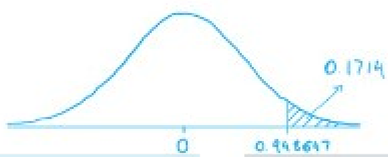
b) i) Standardised normal variable

$Z \sim N(0, 1) \longrightarrow z = \frac{x - \mu}{\sigma}$ (in formula booklet)

Inverse normal function on your GDC.

Area = 0.1714 $\mu = 0$ $\sigma = 1$

$P(Z > 0.948647) = 0.1714$



$0.948647 = \frac{45.37 - 45}{\sigma}$

$\therefore \sigma = 0.390029...$

$\boxed{\sigma = 0.390 \text{ (3sf)}}$

ii) $X \sim N(45, 0.390^2)$

$P(X < 44.5) = 0.09992...$

$\boxed{P(X < 44.5) = 0.0999 \text{ (3sf)}}$

c) Binomial distribution (in formula booklet)

$E(X) = np$ $\quad Var(X) = np(1-p)$

$Y \sim B(15, 0.0999)$

$E(Y) = 15 \times 0.09992$
$= 1.4989...$

$\boxed{E(Y) = 1.50 \text{ (3sf)}}$

d) $Y \sim B(15, 0.0999)$

$P(Y=1) = 0.343287...$

$\boxed{P(Y=1) = 0.343 \text{ (3sf)}}$

e) Conditional probability

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ (in formula booklet)

$P(A \cap B)$

$P(44.63 < X < 45.37) = 0.6572$ (part (a))

$P(B)$

$P(X > 44.5) = 1 - P(X < 44.5)$

$P(X > 44.5) = 0.90007...$

Apply formula

$P(A \mid B) = \dfrac{0.6572}{0.90007}$

$P(A \mid B) = 0.73016...$

$\boxed{P(A \mid B) = 0.730 \text{ (3sf)}}$

Question 5

a) $f(x) = 0$ at A and B

$0 = -k(x^2 - 14x + 24)$

$0 = x^2 - 14x + 24$

$0 = (x-2)(x-12)$

$x = 2 \qquad x = 12$

$\boxed{A(2, 0) \quad \text{and} \quad B(0, 12)}$

b) Vertex $= (7, 8)$, by symmetry

$f(7) = 8$

$8 = -k((7)^2 - 14(7) + 24)$

$8 = 25k$

$\boxed{k = \frac{8}{25}}$

c) Integral of x^n (in formula booklet)

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\int f(x)\,dx = \int_2^{12} -\frac{8}{25}(x^2 - 14x + 24)\,dx$$

$$= -\frac{8}{25}\int_2^{12}(x^2 - 14x + 24)\,dx$$

$$= -\frac{8}{25}\left[\frac{1}{3}x^3 - 7x^2 + 24x\right]_2^{12}$$

$$= -\frac{8}{25}\left[\left(\frac{1}{3}(12)^3 - 7(12)^2 + 24(12)\right) - \left(\frac{1}{3}(2)^3 - 7(2)^2 + 24(2)\right)\right]$$

$$= -\frac{8}{\cancel{25}}\left(-\frac{\cancel{500}^{\,20}}{3}\right)$$

$$= \frac{-8 \times -20}{3}$$

$$\boxed{= \frac{160}{3}\ \text{cm}^2}$$

d) Shaded area $= (10 \times 14) - \frac{160}{3}$

Shaded area $= \frac{260}{3}$ cm^2

Convert to g/cm^3

1060 kg/m^3 $= 1.06$ g/cm^3

mass = density × volume

Let l be the length.

$$2067 = 1.06 \times \frac{260}{3} l$$

$$\boxed{l = 22.5 \text{ cm}}$$

Question 6

a) $-1 \leq \cos(k\pi t) \leq 1$

$h(t)_{max} = 5.59 + 3.6$
$= 9.19 \text{ m}$

$h(t)_{min} = 5.59 - 3.6$
$= 1.99 \text{ m}$

i) $AC = \frac{1}{2}\left(h(t)_{max} - h(t)_{min}\right)$

$AC = \frac{1}{2}(9.19 - 1.99)$

$\boxed{AC = 3.6 \text{ m}}$

ii) $OC = AC + h(t)_{min}$

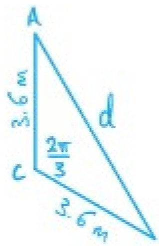
$OC = 3.6 + 1.99$

$\boxed{OC = 5.59 \text{ m}}$

b) Method 1: Cosine rule

$c^2 = a^2 + b^2 - 2ab\cos C$ (in formula booklet)

$a = b = 3.6 \qquad C = \frac{2\pi}{3}$

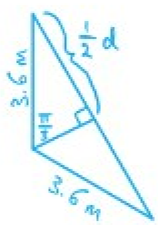


$$d = \sqrt{(3.6)^2 + (3.6)^2 - 2(3.6)(3.6)\cos\left(\frac{2\pi}{3}\right)}$$

$$\boxed{d = \frac{18\sqrt{3}}{5}\text{ m or } 6.24\text{ m (3sf)}}$$

Method 2: Isosceles triangle + SOHCAHTOA

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \frac{\frac{1}{2}d}{3.6}$$



$$d = \cancel{2} \times 3.6 \times \frac{\sqrt{3}}{\cancel{2}}$$

$$d = 3.6 \times \sqrt{3}$$

$$\boxed{d = \frac{18\sqrt{3}}{5}\text{ m or } 6.24\text{ m (3sf)}}$$

c) 75 rpm = 0.8 s per revolution

$k\pi(0) = 0$

$k\cancel{\pi}(0.8) = 2\cancel{\pi}$

$k = \frac{2}{0.8}$

$$\boxed{k = 2.5}$$

d) $h(t)_{min} = 1.99m$

Paul's height $= 1.96m$

Paul is safe if his height $<$ $h(t)_{min}$.

$1.96 < 1.99$

$\boxed{\therefore \text{ Paul is safe.}}$

N.B the argument can be made that 0.03 m is not enough clearance for Paul to be safe.

e) $\tan O\hat{Q}C = \frac{3}{2} \tan O\hat{P}C$ $\left(\tan\theta = \frac{opp}{adj}\right)$

$\frac{OC}{OQ} = \frac{3}{2}\left(\frac{OC}{OP}\right)$

$\frac{OQ}{OC} = \frac{2OP}{3OC}$

$OQ = \frac{2OP}{3\cancel{OC}} \times \cancel{OC}$ $(OP = 9.69 \text{ m})$

$OQ = \frac{2}{3}(9.69)$

$OQ = 6.46m$

$PQ = OP - OQ$

$PQ = 9.69 - 6.46$

$\boxed{PQ = 3.23m}$

f) Method 1

$$A_{APQ} = A_{AOP} - A_{AOQ}$$

$$A_{APQ} = \frac{1}{2}(9.19)(9.69) - \frac{1}{2}(9.19)(6.46)$$

$$A_{APQ} = 14.84185$$

$$\boxed{A_{APQ} = 14.8 \text{ m}^2 \ (3sf)}$$

Method 2

$$A_{APQ} = \frac{1}{2}(AP)(PQ)\sin A\hat{P}Q \qquad \left(\sin A\hat{P}Q = \left(\frac{OA}{AP}\right)\right)$$

$$A_{APQ} = \frac{1}{2}\left(\sqrt{9.19^2 + 9.69^2}\right)(3.23)\left(\frac{9.19}{\sqrt{9.19^2 + 9.69^2}}\right)$$

$$A_{APQ} = 14.84185$$

$$\boxed{A_{APQ} = 14.8 \text{ m}^2 \ (3sf)}$$