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5.9 Advanced Integration

IB Maths - Revision Notes

AA HL

Exam Papers Practice

5.9.1 Integrating Further Functions

As with other problems in integration the results in this revision note may have further uses such as

- evaluating a definite integral
- finding the constant of integration
- finding areas under a curve, between a line and a curve or between two curves

Integrating with Reciprocal Trigonometric Functions

cosec *(cosecant, csc)*, **sec** *(secant)* and **cot** *(cotangent)* are the reciprocal functions of sine, cosine and tangent respectively.

What are the antiderivatives involving reciprocal trigonometric functions?

- $\int \sec^2 x \, dx = \tan x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
- $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
- These are **not** given in the **formula booklet** directly
  - they are listed the other way round as 'standard derivatives'
  - be careful with the negatives in the last two results
  - and remember "+c"!

How do I integrate these if a *linear* function of x is involved?

- All integration rules could apply alongside the results above
- The use of reverse chain rule is particularly common
  - For linear functions the following results can be useful
    - $\int \sec^2(ax+b) \, dx = \frac{1}{a}\tan(ax+b) + \text{c}$
    - $\int \sec(ax+b)\tan(ax+b) \, dx = \frac{1}{a}\sec(ax+b) + \text{c}$
    - $\int \operatorname{cosec}(ax+b)\cot(ax+b) \, dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + \text{c}$
    - $\int \operatorname{cosec}^2(ax+b) \, dx = -\frac{1}{a}\cot(ax+b) + \text{c}$
  - These are **not** in the formula booklet
    - they can be deduced by spotting reverse chain rule
    - they are not essential to remember but can make problems easier

Exam Tip

- Even if you think you have remembered these antiderivatives, always use the formula booklet to double check
    - those squares, negatives and "1 over"'s are easy to get muddled up!
- Remember to use 'adjust' and 'compensate' for reverse chain rule when coefficients are involved

Worked example

The graph of $y = f(x)$ where $f(x) = \int 2\sec^2 5x \, dx$ passes through the point $\left(\frac{\pi}{3}, 0\right)$.

Show that $5y = 2\left(\sqrt{3} + \tan 5x\right)$.

Reverse chain rule is needed

$$\int 2\sec^2 5x \, dx = \underbrace{2 \times \frac{1}{5}}_{\text{'compensate'}} \int \underbrace{5}_{\text{'adjust'}} \sec^2 5x \, dx$$

$$\therefore y = \frac{2}{5}\tan 5x + c \qquad \text{"} \int \sec^2 x \, dx = \tan x + c \text{"}$$

$$\text{At } x = \frac{\pi}{3},\ y = 0, \quad 0 = \frac{2}{5}\overset{-\sqrt{3}}{\cancel{\tan \frac{5\pi}{3}}} + c$$

$$c = \frac{2\sqrt{3}}{5}$$

$$\therefore y = \frac{2}{5}\tan 5x + \frac{2}{5}\sqrt{3}$$

$$y = \frac{2}{5}\left(\tan 5x + \sqrt{3}\right)$$

$$\boxed{\therefore 5y = 2\left(\sqrt{3} + \tan 5x\right)}$$

Integrating with Inverse Trigonometric Functions

arcsin, **arccos** and **arctan** are (one-to-one) functions defined as the inverse functions of sine, cosine and tangent respectively.

What are the antiderivatives involving the inverse trigonometric functions?

- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + c$
- Note that the antiderivative involving $\arccos x$ would arise from
$$\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + c$$
  - However, the negative can be treated as a coefficient of -1 and so
$$\int -\frac{1}{\sqrt{1-x^2}} \, dx = -\int \frac{1}{\sqrt{1-x^2}} \, dx = -\arcsin x + c$$
  - Similarly,
$$\int \frac{1}{\sqrt{1-x^2}} \, dx = -\int -\frac{1}{\sqrt{1-x^2}} \, dx = -\arccos x + c$$
- Unless a question requires otherwise, stick to the first two results
- These are listed in the **formula booklet** the other way round as 'standard derivatives'
- For the antiderivative involving $\arctan x$, note that $(1+x^2)$ is the same as (x^2+1)

How do I integrate these expressions if the denominator is not in the correct form?

- Some problems involve integrands that look very **similar** to the above
  - but the denominators start with a number other than one
  - there are three particular cases to consider
- The first two cases involve denominators of the form $a^2 \pm (bx)^2$ (with or without the square root!)
  - In the case $\boldsymbol{b=1}$ (i.e. denominator of the form $a^2 \pm x^2$) there are two standard results
    - $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$

- $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + c, \quad |x| < a$
  - Both of these **are** given in the **formula booklet**
  - Note in the first result, $a^2 + x^2$ could be written $x^2 + a^2$
- In cases where $b \neq 1$ then the integrand can be rewritten by taking a **factor** of a^2
  - the factor will be a constant that can be taken outside the integral
  - the remaining denominator will then start with 1
  - e.g. $9 + 4x^2 = 9\left(1 + \frac{4}{9}x^2\right) = 9\left(1 + \left(\frac{2}{3}x\right)^2\right)$

- The third type of problem occurs when the denominator has a (three term) quadratic
  - i.e. denominators of the form $ax^2 + bx + c$
    (a rearrangement of this is more likely)
    - the integrand can be rewritten by **completing the square**
    - e.g. $5 - x^2 + 4x = 5 - (x^2 + 4x) = 5 - [(x+2)^2 - 4] = 9 - (x+2)^2$
      This can then be dealt with like the second type of problem above with "x" replaced by "$x + 2$"
    - This works since the derivative of $x + 2$ is the same as the derivative of x
      There is essentially no reverse chain rule to consider

Exam Tip

- Always start integrals involving the inverse trig functions by rewriting the denominator into a recognisable form
  - The numerator and/or any constant factors can be dealt with afterwards, using 'adjust' and 'compensate' if necessary

Worked example

a) Find $\int \frac{1}{9+x^2}\,\mathrm{d}x$.

The denominator is of the form a^2+x^2 so use the result from the formula booklet: "$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$"

$$\boxed{\therefore \int \frac{1}{9+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+c}$$

$9=3^2$

b) Find $\int \frac{1}{\sqrt{5-x^2+4x}}\,\mathrm{d}x$.

The denominator is a three term quadratic so complete the square

$$5-x^2+4x = 5-[x^2-4x]$$
$$= 5-[(x-2)^2-4]$$
$$= 9-(x-2)^2$$

Now write the integral into a recognisable form

$$I = \int \frac{1}{\sqrt{5-x^2+4x}}\,dx = \int \frac{1}{\sqrt{9-(x-2)^2}}\,dx$$

Then use a slight adaption to the result from the formula booklet "$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+c$"

$$\boxed{\therefore I = \arcsin\left(\frac{x-2}{3}\right)+c}$$

Exam Papers Practice

Integrating Exponential & Logarithmic Functions

Exponential functions have the general form $y = a^x$. Special case: $y = \mathrm{e}^x$.

Logarithmic functions have the general form $y = \log_a x$. Special case: $y = \log_{\mathrm{e}} x = \ln x$.

What are the antiderivatives of exponential and logarithmic functions?

- Those involving the special cases have been met before
  - $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$
  - $\int \frac{1}{x} \, \mathrm{d}x = \ln |x| + c$
  - These are given in the **formula booklet**
- Also
  - $\int a^x \, \mathrm{d}x = \frac{1}{\ln a} a^x + c$
  - This is also given in the **formula booklet**
- By reverse chain rule
  - $\int \frac{1}{x \ln a} \, \mathrm{d}x = \log_a |x| + c$
  - This is **not** in the formula booklet
    - but the derivative of $\log_a x$ is given
- There is also the reverse chain rule to look out for
  - this occurs when the numerator is (almost) the derivative of the denominator
  - $\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln |\, f(x)\,| + c$

How do I integrate exponentials and logarithms with a *linear* function of x involved?

- For the special cases involving e and $\ln$
  - $\int \mathrm{e}^{ax+b} \, \mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax+b} + \mathrm{c}$
  - $\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a} \ln |\, ax + b| + c$
- For the general cases

- $\int a^{px+q}\,\mathrm{d}x = \frac{1}{p\ln a}a^{px+q} + c$
- $\int \frac{1}{(px+q)\ln a}\,\mathrm{d}x = \frac{1}{p}\log_a|px+q| + c$

- These four results are **not** in the formula booklet but all can be derived using 'adjust and compensate' from **reverse chain rule**

Exam Tip

- Remember to always use the modulus signs for logarithmic terms in the antiderivative
  - Once it is deduced that $g(x)$ in $\ln|g(x)|$, say, is guaranteed to be positive, the modulus signs can be replaced with brackets

Worked example

a) Show that $\int_1^2 4^x\,\mathrm{d}x = \frac{6}{\ln 2}$.

From the formula booklet, "$\int a^x\,dx = \frac{1}{\ln a}a^x + c$"

$$\therefore \int_1^2 4^x\,dx = \left[\frac{1}{\ln 4}4^x\right]_1^2$$

$$= \frac{16}{\ln 4} - \frac{4}{\ln 4}$$

$$= \frac{12}{\ln 4}$$

$$= \frac{12}{2\ln 2}$$

$\ln 4 = \ln 2^2 = 2\ln 2$

$$\boxed{\therefore \int_1^2 4^x\,dx = \frac{6}{\ln 2}}$$

b) Find $\int \frac{1}{(2x-1)\ln 3}\,\mathrm{d}x$.

The result $\displaystyle\int \frac{1}{(px+q)\ln a}\,dx = \frac{1}{p}\log_a|px+q| + c$

could be used but this is not in the formula booklet.

Alternatively use reverse chain rule with the result " $f(x) = \log_a x, \quad f'(x) = \dfrac{1}{x\ln a}$ " which is given in the formula booklet!

$$\therefore I = \int \frac{1}{(2x-1)\ln 3}\,dx = \underbrace{\frac{1}{2}}_{\text{'compensate'}}\int \frac{\overbrace{2}^{\text{'adjust'}}}{(2x-1)\ln 3}\,dx$$

$$\boxed{\therefore I = \frac{1}{2}\log_3|2x-1| + c}$$

... and '+c'!

remember the modulus signs...

5.9.2 Further Techniques of Integration

Integration by Substitution

What is integration by substitution?

- Integration by substitution is used when an integrand where reverse chain rule is either not obvious or is not spotted
  - in the latter case it is like a "back-up" method for reverse chain rule

How do I use integration by substitution?

- For instances where the substitution is not obvious it will be given in a question
  - e.g. Find $\int \cot x \, \mathrm{d}x$ using the substitution $u = \sin x$
- Substitutions are usually of the form $u = g(x)$
  - in some cases $u^2 = g(x)$ and other variations are more convenient
    - as these would not be obvious, they would be given in a question
  - if need be, this can be rearranged to find x in terms of u
- Integration by substitution then involves rewriting the integral, including "$\mathrm{d}x$" in terms of u

STEP 1
Name the integral to save rewriting it later
Identify the given substitution $u = g(x)$

STEP 2
Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and rearrange into the form $f(u)\,\mathrm{d}u = g(x)\,\mathrm{d}x$ such that (some of) the integral can be rewritten in terms of u

STEP 3
If limits are involved, use $u = g(x)$ to change them from x values to u values

STEP 4
Rewrite the integral so everything is in terms of u rather than x
This is the step when it may become apparent that x is needed in terms of u

STEP 5
Integrate with respect to u and either rewrite in terms of x or apply the limits using their u values

- For quotients the substitution usually involves the denominator
- It may be necessary to use 'adjust and compensate' to deal with any coefficients in the integrand
- Although $\frac{du}{dx}$ can be treated like a fraction it should be appreciated that this is a 'shortcut' and the maths behind it is beyond the scope of the IB course

Exam Tip

- If a substitution is not given in a question, it is usually because it is obvious
  - If you can't see anything obvious, or you find that your choice of substitution doesn't reduce the integrand to something easy to integrate, consider that it may not be a substitution question

Worked example

Use the substitution $u = (1 + 2x)$ to evaluate $\int_0^1 x(1+2x)^7 \, dx$.

STEP 1: Name the integral, identify the substitution

$$I = \int_0^1 x(1+2x)^7 \, dx$$

$u = 1 + 2x$

STEP 2: Find $\frac{du}{dx}$ and rearrange

$$\frac{du}{dx} = 2$$

$$\frac{1}{2} du = dx$$

STEP 3: Change limits from x values to u values

$x = 0, \quad u = 1 + 2(0) = 1$

$x = 1, \quad u = 1 + 2(1) = 3$

STEP 4: Rewrite the integral, find x in terms of u

$$I = \int_1^3 \frac{1}{2}(u-1)u^7 \times \frac{1}{2} du = \frac{1}{4}\int_1^3 (u^8 - u^7) \, du$$

x in terms of u
$u = 1 + 2x$
$\therefore x = \frac{1}{2}(u-1)$

STEP 5: Integrate and evaluate

$$I = \frac{1}{4}\left[\frac{u^9}{9} - \frac{u^8}{8}\right]_1^3$$

$$I = \frac{1}{4}\left[\left(\frac{3^9}{9} - \frac{3^8}{8}\right) - \left(\frac{1^9}{9} - \frac{1^8}{8}\right)\right]$$

$$\boxed{\therefore I = \frac{6151}{18}}$$

Integration by Parts

What is integration by parts?

- Integration by parts is generally used to integrate the product of two functions
  - however reverse chain rule and/or substitution should be considered first
    - e.g. $\int 2x\cos(x^2)\ dx$ can be solved using reverse chain rule or the substitution $u = x^2$
  - Integration by parts is essentially 'reverse product rule'
    - whilst every product can be differentiated, not every product can be integrated (analytically)

What is the formula for integration by parts?

- $\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx$
- This is given in the **formula booklet** alongside its alternative form $\int u\ dv = uv - \int v\ du$

How do I use integration by parts?

- For a given integral u and $\frac{dv}{dx}$ (rather than u and v) are assigned functions of x
- Generally, the function that becomes simpler when differentiated should be assigned to u
- There are various stages of integrating in this method
  - only one overall constant of integration ("+c") is required
  - put this in at the last stage of working
  - if it is a definite integral then "+c" is not required at all

STEP 1
Name the integral if it doesn't have one already!
This saves having to rewrite it several times – I is often used for this purpose.

e.g. $I = \int x\sin x\ dx$

STEP 2

Assign u and $\frac{dv}{dx}$.

Differentiate u to find $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to find v

e.g. $u = x \qquad v = -\cos x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sin x$

STEP 3
Apply the integration by parts formula

e.g. $I = -x\cos x - \int -\cos x \, dx$

STEP 4

Work out the second integral, $\int v \frac{du}{dx} \, dx$

Now include a "+c" (unless definite integration)
e.g. $I = -x\cos x + \sin x + c$

STEP 5
Simplify the answer if possible or apply the limits for definite integration
e.g. $I = \sin x - x\cos x + c$

- In trickier problems other rules of differentiation and integration may be needed
  - chain, product or quotient rule
  - reverse chain rule, substitution

Can integration by parts be used when there is only a single function?

- Some single functions (non-products) are awkward to integrate directly
  - e.g. $y = \ln x$, $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$
- These can be integrated using parts however
  - rewrite as the product '$1 \times f(x)$' and choose $u = f(x)$ and $\frac{dv}{dx} = 1$
  - 1 is easy to integrate and the functions above have standard derivatives listed in the formula booklet

Exam Tip

- If $\ln x$ or one of the inverse trig functions are one of the functions involved in the product then these should be assigned to "u" when applying parts
  - They are (realtively) easy to differentiate (to find u') but are awkward to integrate

Worked example

a) Find $\int 5xe^{3x}\,dx$.

STEP 1: Name the integral

$$I = \int 5xe^{3x}\,dx = 5\int xe^{3x}\,dx$$

STEP 2: Assign u and v'

Find u' and v

$u = x$ $v = \frac{1}{3}e^{3x}$ (reverse chain rule)

$u' = 1$ $v' = e^{3x}$

x becomes simpler when differentiated

STEP 3: Apply the integration by parts formula

$$I = 5\left[\frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx\right]$$

STEP 4: Work out the second integral

$$I = 5\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c\right]$$

include "+c" at last working stage

reverse chain rule

STEP 5: Simplify

$$\boxed{I = \frac{5}{9}e^{3x}(3x-1) + c}$$

b) Show that $\int 8x\ln x\,dx = 2x^2(1 + \ln x^2) + c$.

STEP 1: Name the integral

$$I = \int 8x \ln x \, dx$$

STEP 2: Assign u and v' – as $\ln$ is involved, $u = \ln x$

Find u' and v

$u = \ln x \qquad v = 4x^2$

$u' = \frac{1}{x} \qquad v' = 8x$

STEP 3: Apply the integration by parts formula

$$I = 4x^2 \ln x - \int 4x^2 \times \frac{1}{x} \, dx = 4x^2 \ln x - \int 4x \, dx$$

STEP 4: Work out the second integral, include "$+c$" at this stage

$$I = 4x^2 \ln x - 2x^2 + c$$

STEP 5: Simplify

$$I = 2x^2(2\ln x - 1) + c$$

$$\boxed{\therefore I = 2x^2(\ln x^2 - 1) + c}$$

Repeated Integration by Parts

When will I have to repeat integration by parts?

- In some problems, applying integration by parts still leaves the second integral as a product of two functions of x
  - integration by parts will need to be applied again to the second integral
- This occurs when one of the functions takes more than one derivative to become simple enough to make the second integral straightforward
  - These functions usually have the form $x^2 g(x)$

How do I apply integration by parts more than once?

STEP 1
Name the integral if it doesn't have one already!

STEP 2
Assign u and $\frac{\mathrm{d}v}{\mathrm{d}x}$. Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and v

STEP 3
Apply the integration by parts formula

STEP 4
Repeat STEPS 2 and 3 for the second integral

STEP 5
Work out the second integral and include a "+c" if necessary

STEP 6
Simplify the answer or apply limits

What if neither function ever becomes simpler when differentiating?

- It is possible that integration by parts will end up in a seemingly endless loop
  - consider the product $\mathrm{e}^x \sin x$
  - the derivative of e^x is e^x
    - no matter how many times a function involving e^x is differentiated, it will still involve e^x
  - the derivative of $\sin x$ is $\cos x$
    - $\cos x$ would then have derivative $-\sin x$, and so on
    - no matter how many times a function involving $\sin x$ or $\cos x$ is differentiated, it will still involve $\sin x$ or $\cos x$

- This loop can be trapped by spotting when the second integral becomes identical to (or a multiple of) the original integral
  - naming the original integral (I) at the start helps
  - I then appears twice in integration by parts
    - e.g. $I = g(x) - I$
      where $g(x)$ are parts of the integral not requiring further work
  - It is then straightforward to rearrange and solve the problem
    - e.g. $2I = g(x) + c$
      $I = \frac{1}{2} g(x) + c$

Worked example

a) Find $\int x^2 \cos x \, dx$.

STEP 1: Name the integral

$$I = \int x^2 \cos x \, dx$$

STEP 2: Assign u and v'

Find u' and v

$u = x^2$ $\quad$ $v = \sin x$

$u' = 2x$ $\quad$ $v' = \cos x$

x^2 becomes 'simpler' when differentiated

STEP 3: Apply the integration by parts formula

$$I = x^2 \sin x - 2\int x \sin x \, dx$$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$u = x$ $\quad$ $v = -\cos x$

$u' = 1$ $\quad$ $v' = \sin x$

$$I = x^2 \sin x - 2\left[-x\cos x - \int -\cos x \, dx\right]$$

STEP 5: Work out the second integral now it is straightforward

$$I = x^2 \sin x + 2x\cos x - 2\sin x + c$$

STEP 6: Simplify

$$\boxed{I = (x^2 - 2)\sin x + 2x\cos x + c}$$

b) Find $\int e^x \sin x \, dx$.

STEP 1: Name the integral

$$I = \int e^{x} \sin x \, dx$$

STEP 2: Assign u and v'. Neither function becomes simpler when differentiated. Find u' and v.

$u = e^{x}$ $\quad$ $v = -\cos x$

$u' = e^{x}$ $\quad$ $v' = \sin x$

STEP 3: Apply the integration by parts formula

$$I = -e^{x}\cos x - \int -e^{x}\cos x \, dx = -e^{x}\cos x + \int e^{x}\cos x \, dx$$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$u = e^{x}$ $\quad$ $v = \sin x$

$u' = e^{x}$ $\quad$ $v' = \cos x$

$$I = -e^{x}\cos x + \left[e^{x}\sin x - \int e^{x}\sin x \, dx \right]$$

Spot that this is the same as the original question, i.e. I

STEP 5: "Work out" the second integral, include "+c" at this stage

$$I = e^{x}\sin x - e^{x}\cos x - I + c$$

STEP 6: Simplify

$$2I = e^{x}(\sin x - \cos x) + c$$

$$\boxed{\therefore I = \tfrac{1}{2}e^{x}(\sin x - \cos x) + c_1 \qquad (\text{where } c_1 = \tfrac{1}{2}c)}$$

5.9.3 Integrating with Partial Fractions

Integrating with Partial Fractions

What are partial fractions?

- **Partial fractions** arise when a quotient is rewritten as the **sum** of fractions
  - The process is the opposite of adding or subtracting fractions
- Each partial fraction has a denominator which is a **linear factor** of the quotient's denominator
  - e.g. A quotient with a denominator of $x^2 + 4x + 3$
    - factorises to $(x + 1)(x + 3)$
    - so the quotient will split into two partial fractions
    - one with the (linear) denominator $(x + 1)$
    - one with the (linear) denominator $(x + 3)$

How do I know when to use partial fractions in integration?

- For this course, the denominators of the quotient will be of quadratic form
  - i.e. $f(x) = ax^2 + bx + c$
  - check to see if the quotient can be written in the form $\frac{f'(x)}{f(x)}$
    - in this case, reverse chain rule applies
- If the denominator does not factorise then the **inverse trigonometric functions** are involved

How do I integrate using partial fractions?

STEP 1
Write the quotient in the integrand as the sum of partial fractions
This involves factorising the denominator, writing it as an identity of two partial fractions and using values of x to find their numerators

e.g. $I = \int \frac{1}{x^2 + 4x + 3}\, dx = \int \frac{1}{(x + 1)(x + 3)}\, dx = \frac{1}{2} \int \left(\frac{1}{x + 1} - \frac{1}{x + 3}\right) dx$

STEP 2
Integrate each partial fraction leading to an expression involving the sum of natural logarithms

e.g. $I = \frac{1}{2} \int \left(\frac{1}{x + 1} - \frac{1}{x + 3}\right) dx = \frac{1}{2}[\ln|x + 1| - \ln|x + 3|] + c$

STEP 3
Use the laws of logarithms to simplify the expression and/or apply the limits
(Simplifying first may make applying the limits easier)

e.g. $I = \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + c$

- By rewriting the constant of integration as a logarithm ($c = \ln k$, say) it is possible to write the final answer as a single term

  e.g. $I = \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + \ln k = \ln\sqrt{\left|\frac{x+1}{x+3}\right|} + \ln k = \ln\left(k\sqrt{\left|\frac{x+1}{x+3}\right|}\right)$

Exam Tip

- Always check to see if the numerator can be written as the derivative of the denominator
  - If so then it is reverse chain rule, not partial fractions
  - Use the number of marks a question is worth to help judge how much work should be involved

Worked example

Find $\int \frac{3x+1}{x^2+3x-10}\,dx$.

The integrand is NOT of the form $\frac{f'(x)}{f(x)}$ but the denominator does factorise

STEP 1: Write the quotient as partial fractions

$$\frac{3x+1}{x^2+3x-10} \equiv \frac{A}{x+5} + \frac{B}{x-2}$$

$$3x+1 \equiv A(x-2) + B(x+5)$$

Let $x=2$, $7=7B$, $B=1$

Let $x=-5$, $-14=-7A$, $A=2$

$$\therefore I = \int \frac{3x+1}{x^2+3x-10}\,dx = \int \left(\frac{2}{x+5} + \frac{1}{x-2}\right) dx$$

STEP 2: Integrate the partial fractions

$$I = 2\ln|x+5| + \ln|x-2| + c$$

STEP 3: Simplify using laws of logarithms

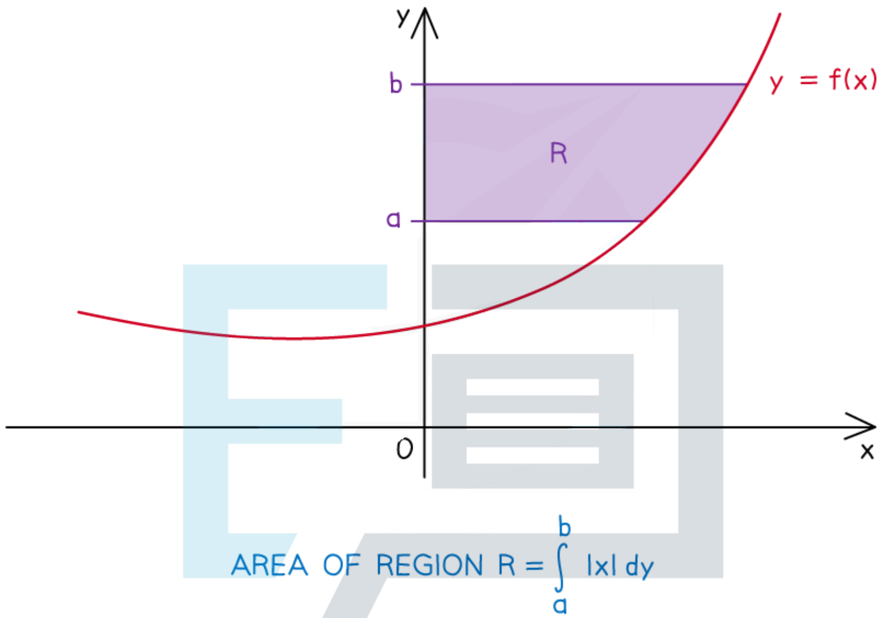
$$I = \ln(x+5)^2 + \ln|x-2| + c$$

$$\boxed{\therefore I = \ln\left|(x+5)^2(x-2)\right| + c}$$

5.9.4 Advanced Applications of Integration

Area Between Curve & y-axis

What is meant by the area between a curve and the y-axis?



- The area referred to is the region bounded by
  - the graph of $y = f(x)$
  - the y-axis
  - the horizontal line $y = a$
  - the horizontal line $y = b$
- The exact area can be found by evaluating a definite integral
- The graph of $y = f(x)$ could be a straight line
  - using basic shape area formulae may be easier than integration
    - e.g. area of a trapezium: $A = \frac{1}{2}h(a+b)$

How do I find the area between a curve and the y-axis?

- Use the formula

$$A = \int_a^b |x|\ \mathrm{d}x$$

  - This is given in the **formula booklet**
  - The function is normally given in the form $y = f(x)$
    - so will need rearranging into the form $x = g(y)$

- a and b may not be given directly as could involve the x-axis ($y = 0$) and/or a root of $x = g(y)$
  - use a GDC to plot the curve, sketch it and highlight the area to help

STEP 1
Identify the limits a and b
Sketch the graph of $y = f(x)$ or use a GDC to do so, especially if a and b are not given directly in the question

STEP 2
Rearrange $y = f(x)$ into the form $x = g(y)$
This is similar to finding the inverse function $f^{-1}(x)$

STEP 3
Evaluate the formula to evaluate the integral and find the area required
If using a GDC remember to include the modulus (|...|) symbols around x

- In trickier problems some (or all) of the area may be 'negative'
  - this will be any area that is left of the y-axis (negative x-values)
  - $|x|$ makes such areas 'positive'
    - a GDC will apply '$|x|$' automatically as long as the |...| are included
    - otherwise, to apply '$|x|$', split the integral into positive and negative parts; write an integral and evaluate each part separately and add the modulus of each part together to give the total area

Exam Tip

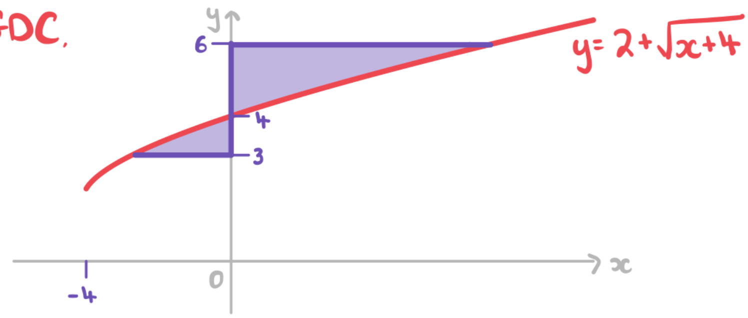
- Sketch and/or use your GDC to help visualise what the problem looks like

Worked example

Find the area enclosed by the curve with equation $y = 2 + \sqrt{x + 4}$, the y-axis and the horizontal lines with equations $y = 3$ and $y = 6$.

STEP 1: Identify limits, sketch graph/use GDC

From GDC.



STEP 2: Rearrange $y=f(x)$ into $x=g(y)$

$y = 2+\sqrt{x+4}$

$x = (y-2)^2 - 4 = y^2 - 4y + 4 - 4$

$x = y^2 - 4y$

STEP 3: Evaluate integral to find area

As some area 'is' negative, split the integral

$$A = -\int_3^4 (y^2-4y)\,dy + \int_4^6 (y^2-4y)\,dy$$

From GDC/sketch this area is 'negative'

If using GDC you can still do this in one go:

$$\int_3^6 |y^2-4y|\,dy$$

$$\therefore A = \left[\frac{y^3}{3} - 2y^2\right]_4^6 - \left[\frac{y^3}{3} - 2y^2\right]_3^4$$

$$A = \left[(72-72) - \left(\frac{64}{3} - 32\right)\right] - \left[\left(\frac{64}{3} - 32\right) - (9-18)\right]$$

$$A = \frac{32}{3} - -\frac{5}{3}$$

$$\boxed{\therefore A = \frac{37}{3} \text{ square units}}$$

Volumes of Revolution Around x-axis

What is a volume of revolution around the x-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the x-axis
- The **volume** of **revolution** is the volume of this solid
- Be careful – the 'front' and 'back' of this solid are flat
  - they were created from straight (vertical) lines
  - 3D sketches can be misleading

How do I solve problems involving the volume of revolution around x-axis?

- Use the formula

$$V = \pi \int_a^b y^2 \, dx$$

- This is given in the **formula booklet**
- y is a function of x
- $x = a$ and $x = b$ are the equations of the (vertical) lines bounding the area
  - If $x = a$ and $x = b$ are not stated in a question, the boundaries could involve the y-axis ($x = 0$) and/or a root of $y = f(x)$
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1
Identify the limits a and b
Sketching the graph of $y = f(x)$ or using a GDC to do so is helpful, especially when a and b are not given directly in the question

STEP 2
Square y

STEP 3
Use the formula to evaluate the integral and find the volume of revolution
An answer may be required in exact form

Exam Tip

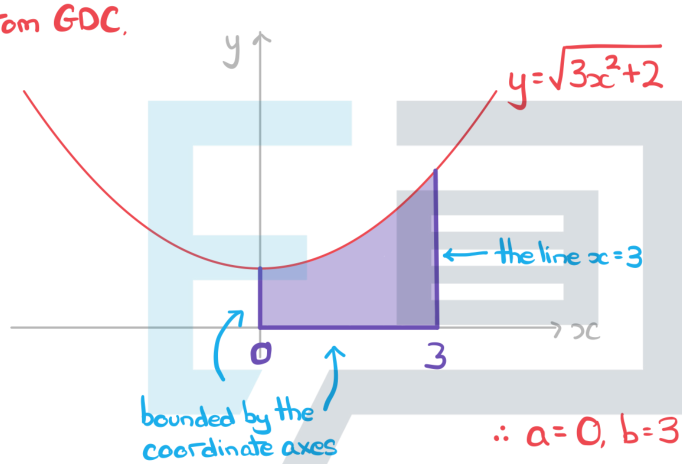
- If the given function involves a square root(s), problems can seem quite daunting
  - However, this is often deliberate, as the square root will be squared when applying the Volume of Revolution formula, and should leave the integrand as something more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise and make progress with problems

Worked example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \sqrt{3x^2 + 2}$, the coordinate axes and the line $x = 3$ by 2π radians around the x-axis. Give your answer as an exact multiple of π.

STEP 1: Identify limits, sketch graph/use GDC

From GDC,



$\therefore a = 0,\ b = 3$

STEP 2: Square y

$y^2 = \left(\sqrt{3x^2+2}\right)^2 = 3x^2 + 2$

STEP 3: Find the volume

$$V = \pi \int_0^3 (3x^2 + 2)\,dx = \pi\left[x^3 + 2x\right]_0^3$$
$$= \pi(27 + 6)$$

$\boxed{\therefore V = 33\pi \text{ cubic units}}$

Volumes of Revolution Around y-axis

What is a volume of revolution around the y-axis?

- Very similar to above, this is a **solid** of **revolution** which is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the y-axis
- The **volume** of **revolution** is the volume of this solid

How do I solve problems involving the volume of revolution around y-axis?

- Use the formula

$$V = \pi \int_a^b x^2 \, \mathrm{d}y$$

- This is given in the **formula booklet**
- The function is usually given in the form $y = f(x)$
  - so will need rearranging into the form $x = g(y)$
- a and b may not be given directly as could involve the x-axis ($y = 0$) and/or a root of $x = g(y)$
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful

STEP 1
Identify the limits a and b
Sketching the graph of $y = f(x)$ or using a GDC to do so is helpful, especially if a and b are not given directly in the question

STEP 2
Rearrange $y = f(x)$ into the form $x = g(y)$
This is similar to finding the inverse function $f^{-1}(x)$

STEP 3
Square x

STEP 4
Use the formula to evaluate the integral and find the volume of revolution
An answer may be required in exact form

Exam Tip

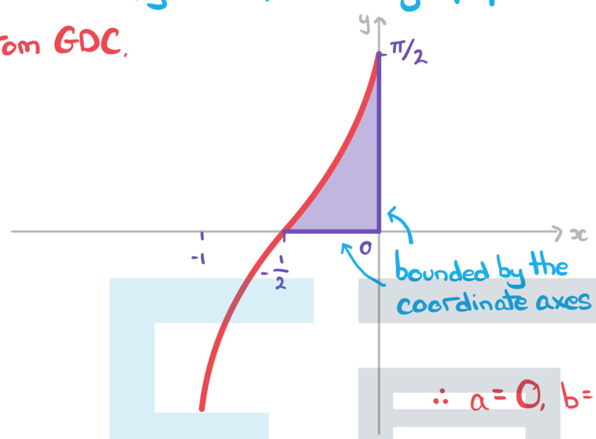
- If the given function involves a square root, problems can seem quite daunting
  - This is often deliberate, as the square root will be squared when applying the Volume of Revolution formula and the integrand will then become more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise the problem and make progress

Worked example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \arcsin(2x+1)$ and the coordinate axes by 2π radians around the y-axis. Give your answer to three significant figures.

STEP 1: Identify limits, sketch graph/use GDC

From GDC,



$\therefore\ a = 0,\ b = \pi/2$

STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = \arcsin(2x+1)$

$\sin y = 2x + 1$

$x = \frac{1}{2}(\sin y - 1)$

STEP 3: Square x

$x^2 = \frac{1}{4}(\sin y - 1)^2$

STEP 4: Find the volume

$$V = \pi \int_0^{\pi/2} \frac{1}{4}(\sin y - 1)^2 \, dy$$

As this is awkward, use your GDC but

- your GDC will expect the integrand in terms of x
- remember π !

$V = 0.279\ 754\ \ldots$

$\boxed{\therefore V = 0.280 \text{ cubic units } (3 \text{ s.f.})}$

5.9.5 Modelling with Volumes of Revolution

The volume of the solid of revolution formed by rotating an area through 2π radians around the x-axis is $V = \pi \int_a^b y^2 \, dx$, and for the y-axis is $V = \pi \int_a^b x^2 \, dy$. These are both given in the **formula booklet**.

Adding & Subtracting Volumes

When would volumes of revolution need to be added or subtracted?

- The 'curve' boundary of an area may consist of **more than one** function of x
  - For example
    - the 'curve' boundary from $x = 0$ to $x = 3$ is $y = f(x)$
    - the 'curve' boundary from $x = 3$ to $x = 6$ is $y = g(x)$
  - So the **total volume** would be $V = \pi \int_0^3 [f(x)]^2 \, dx + \pi \int_3^6 [g(x)]^2 \, dx$
- The solid of revolution may have a 'hole' in it
  - e.g. a 'toilet roll' shape would be the **difference** of two cylindrical volumes

How do I know whether to add or subtract volumes of revolution?

- When the **area** to be **rotated** around the x-axis has more than one function defining its boundary it can be trickier to tell whether to **add** or **subtract volumes** of **revolution**
  - It will depend on the **nature** of the **functions** and their **points** of **intersection**
  - With help from a GDC, sketch the graph of the functions and highlight the area required

How do I solve problems involving adding or subtracting volumes of revolution?

- Visualising the solid created becomes increasingly useful (but also trickier) for shapes generated by separate volumes of revolution
  - Continue trying to sketch the functions and their solids of revolution to help

STEP 1
Identify the functions $(y = f(x),\ y = g(x),\ \ldots)$ involved in generating the volume
Determine whether the separate volumes will need to be added or subtracted
Identify the limits for each volume involved
Sketching the graphs of $y = f(x)$ and $y = g(x)$, or using a GDC to do so, is helpful, especially when the limits are not given directly in the question

STEP 2
Square y for all functions $([f(x)]^2,\ [g(x)]^2,\ \ldots)$
This step is not essential if a GDC can be used to calculate integrals and an exact answer is not required.

STEP 3
Use the appropriate volume of revolution formula for each part, evaluate the definite integral and add or subtract as necessary
The answer may be required in exact form

Exam Tip

- A sketch of the graph, limits, etc is always helpful, whether one has been given in the question or not
  - Use your GDC where possible

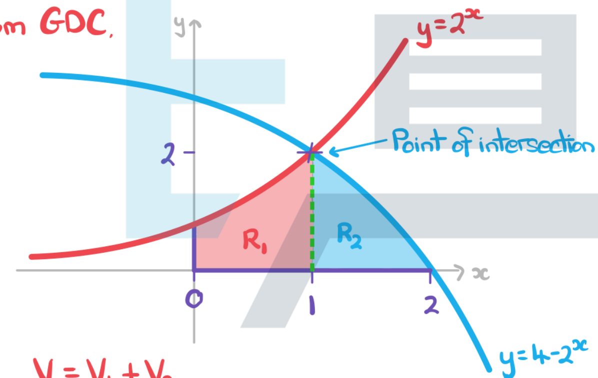
Worked example

Find the volume of revolution of the solid formed by rotating the region enclosed by the positive coordinate axes and the graphs of $y = 2^x$ and $y = 4 - 2^x$ by 2π radians around the x-axis. Give your answer to three significant figures.

STEP 1: Identify functions, limits and whether to add or subtract

Use GDC to help sketch the graphs

From GDC,



$2^x = 4 - 2^x$
$2(2^x) = 4$
$2^x = 2$
$x = 1, \ y = 2^1 = 2$

$V = V_1 + V_2$

For R_1, $a=0$, $b=1$

For R_2, $a=1$, $b=2$

STEP 2: Square all functions - this step is not required in this question

STEP 3: Use formula for each part, evaluate and add

$$V = \pi \int_0^1 (2^x)^2 \, dx + \pi \int_1^2 (4 - 2^x)^2 \, dx$$

Use your GDC to evaluate - to avoid typing errors evaluate each integral separately, store in memory, then add

$V = 6.798\,540\ldots + 4.941\,881\ldots = 11.740\ldots$

$\boxed{\therefore V = 11.7 \text{ cubic units } (3 \text{ s.f.})}$

Modelling with Volumes of Revolution

What is meant by modelling volumes of revolution?

- Many everyday objects such as buckets, beakers, vases and lamp shades can be modelled as a **solid of revolution**
- The volume of revolution of the solid can then be calculated
- An object that would usually stand **upright** can be **modelled horizontally** so its **volume of revolution** can be found

What modelling assumptions are there with volumes of revolution?

- The solids formed are usually the main shape of the body of the object
  - For example, the handle on a bucket would not be included
- The thickness of the solid is negligible relative to the size of the object
  - thickness will depend on the purpose of the object and the material it is made from

How do I solve modelling problems with volumes of revolution?

- Visualising and sketching the solid formed can help with starting problems
- Familiarity with applying the volume of revolution fomulae
  - around the x-axis: $V = \int_a^b y^2 \, \mathrm{d}x$
  - around the y-axis: $V = \int_a^b x^2 \, \mathrm{d}y$
- The volume of revolution may involve adding or subtracting partial volumes
- Questions may ask related questions in context
  - g. A question about a bucket may ask about its **capacity**
    - this would be measured in litres
    - so a conversion of units may be required
    - (100 cm^3 = 1 litre)

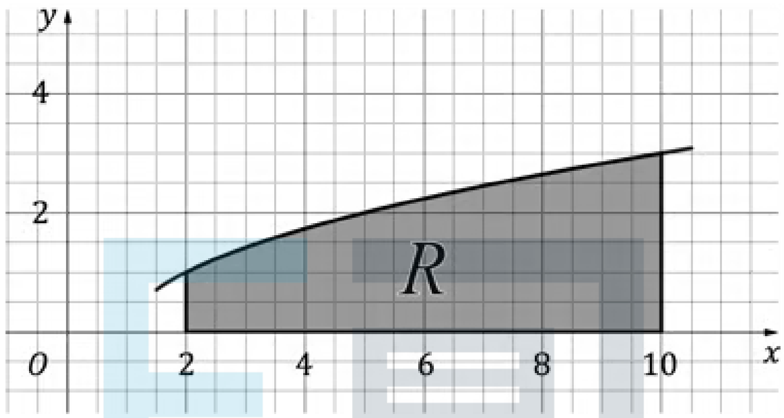
Exam Tip

- Remember to answer questions directly
  - In modelling scenarios, interpretation is often needed after finding the 'final answer'
- Modelling questions often ask about assumptions, criticisms and/or improvements
- Examples
  - it is assumed the thickness of the material an object is made from is negligible
  - a 'smooth' curve may not be a good model if the item is being made from a rough material
  - other things may significantly reduce the volume found and impact conclusions
    - e.g. Stones, plants and decorations placed in an aquarium will reduce the volume of water needed to fill it - and hence the number/size/type of fish it can accommodate may be impacted

Worked example

The diagram below shows the region *R*, which is bounded by the function $y = \sqrt{x-1}$, the lines $x = 2$ and $x = 10$, and the x-axis.

Dimensions are in centimetres.



A mathematical model for a miniature vase is produced by rotating the region *R* through 2π radians around the x-axis.

Find the volume of the miniature vase, giving your answer in litres to three significant figures.

Exam Papers Practice

STEP 1 Identify limits

$a = 2$

$b = 10$

STEP 2 Square y

$y^2 = \left(\sqrt{x-1}\right)^2 = x - 1$

STEP 3 Evaluate the integral

$$V = \pi \int_2^{10} (x-1)\, dx = \pi\left[0.5x^2 - x\right]_2^{10}$$

$$= \pi\left[(50-10) - (2-2)\right]$$

$$= 40\pi$$

Now we need to interpret this in the context of the miniature vase

$V = 40\pi \text{ cm}^3$

$V = \dfrac{40\pi}{1000}$ litres $\qquad 1000\text{cm}^3 = 1$ litre

$V = 0.125\,663\ldots$

$\boxed{\text{Volume of the miniature vase is } 0.126 \text{ litres (3 s.f.)}}$