DP IB Maths: AI HL

5.7 Further Differential Equations

Contents

5.7.1 Coupled Differential Equations

Solving Coupled Differential Equations

How do I write a system of coupled differential equations in matrix form?

- The coupled differential equations considered in this part of the course will be of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

  - $a, b, c, d \in \mathbb{R}$ are constants whose precise value will depend on the situation being modelled
    - In an exam question the values of the constants will generally be given to you
- This system of equations can also be represented in matrix form:

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- It is usually more convenient, however, to use the 'dot notation' for the derivatives:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- This can be written even more succinctly as $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}$
  - Here $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

How do I find the exact solution for a system of coupled differential equations?

- The exact solution of the coupled system $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}$ depends on the **eigenvalues** and **eigenvectors** of the matrix of coefficients $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
  - The eigenvalues and/or eigenvectors may be given to you in an exam question
  - If they are not then you will need to calculate them using the methods learned in the matrices section of the course
- On the exam you will only be asked to find exact solutions for cases where the two eigenvalues of the matrix are real, distinct, and non-zero

- Similar solution methods exist for non-real, non-distinct and/or non-zero eigenvalues, but you don't need to know them as part of the IB AI HL course
- Let the eigenvalues and corresponding eigenvectors of matrix $\boldsymbol{M}$ be λ_1 and λ_2, and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, respectively
  - Remember from the definition of eigenvalues and eigenvectors that this means that $\boldsymbol{M}\boldsymbol{p}_1 = \lambda_1 \boldsymbol{p}_1$ and $\boldsymbol{M}\boldsymbol{p}_2 = \lambda_2 \boldsymbol{p}_2$
- The exact solution to the system of coupled differential equations is then

$$\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$$

  - This solution formula is in the exam formula booklet
  - $A,\ B \in \mathbb{R}$ are constants (they are essentially constants of integration of the sort you have when solving other forms of differential equation)
- If initial or boundary conditions have been provided you can use these to find the precise values of the constants A and B
  - Finding the values of A and B will generally involve solving a set of simultaneous linear equations (see the worked example below)

Worked example

The rates of change of two variables, x and y, are described by the following system of coupled differential equations:

$$\frac{dx}{dt} = 4x - y$$
$$\frac{dy}{dt} = 2x + y$$

Initially $x = 2$ and $y = 1$.

Given that the matrix $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ has eigenvalues of 3 and 2 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the exact solution to the system of coupled differential equations.

Exact solution for coupled linear differential equations	$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$

From exam formula booklet

$$\underline{x} = Ae^{3t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{2t}\begin{pmatrix}1\\2\end{pmatrix}$$

Put eigenvalues and eigenvectors into the solution formula

At $t=0$, $\underline{x} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

Use initial condition to find values of A and B

So $Ae^0\begin{pmatrix}1\\1\end{pmatrix} + Be^0\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$$\begin{pmatrix}A+B\\A+2B\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

Simultaneous equations

$\Rightarrow A = 3,\ B = -1$

$$\underline{x} = 3e^{3t}\begin{pmatrix}1\\1\end{pmatrix} - e^{2t}\begin{pmatrix}1\\2\end{pmatrix}$$

I.e., $x = 3e^{3t} - e^{2t}$

$y = 3e^{3t} - 2e^{2t}$

Your notes

Phase Portraits

What is a phase portrait for a system of coupled differential equations?

- Here we are again considering systems of coupled equations that can be represented in the matrix form $\dot{\boldsymbol{x}} = \boldsymbol{Mx}$, where $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$
- A **phase portrait** is a diagram showing how the values of *x* and *y* change over time
  - On a phase portrait we will usually sketch several typical solution trajectories
  - The precise trajectory that the solution for a particular system will travel along is determined by the initial conditions for the system
- Let λ_1 and λ_2 be the eigenvalues of the matrix $\boldsymbol{M}$
  - The overall nature of the phase portrait depends in large part on the values of λ_1 and λ_2

What does the phase portrait look like when λ_1 and λ_2 are real numbers?

- Recall that for real distinct eigenvalues the solution to a system of the above form is $\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$, where λ_1 and λ_2 are the eigenvalues of $\boldsymbol{M}$ and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the corresponding eigenvectors
  - AI HL only considers cases where λ_1 and λ_2 are distinct (i.e., $\lambda_1 \neq \lambda_2$) and non-zero
- A phase portrait will always include two 'eigenvector lines' through the origin, each one parallel to one of the eigenvectors
  - So if $\boldsymbol{p}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{p}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, for example, then these lines through the origin will have equations $y = 2x$ and $y = -\frac{4}{3}x$, respectively
  - These lines will define two sets of solution trajectories
  - If the eigenvalue corresponding to a line's eigenvector is **positive**, then there will be solution trajectories along the line **away from** the origin in both directions as *t* increases
  - If the eigenvalue corresponding to a line's eigenvector is **negative**, then there will be solution trajectories along the line **towards** the origin in both directions as *t* increases
  - No solution trajectory will ever cross an eigenvector line
- If **both eigenvalues are positive** then all solution trajectories will be directed **away from** the origin as *t* increases
  - In between the 'eigenvector lines' the trajectories as they move away from the origin will all curve to become approximately parallel to the line whose eigenvector corresponds to the larger eigenvalue

- If **both eigenvalues are negative** then all solution trajectories will be directed **towards** the origin as *t* increases
  - In between the 'eigenvector lines' the trajectories will all curve so that at points further away from the origin they are approximately parallel to the line whose eigenvector corresponds to the more negative eigenvalue
    - They will then converge on the other eigenvalue line as they move in towards the origin

- If **one eigenvalue is positive and one eigenvalue is negative** then solution trajectories will generally start by heading in towards the origin before curving to head out away again from the origin as *t* increases
  - In between the 'eigenvector lines' the solution trajectories will all move in towards the origin along the direction of the eigenvector line that corresponds to the negative eigenvalue, before curving away and converging on the eigenvector line that corresponds to the positive eigenvalue as they head away from the origin

What does the phase portrait look like when λ_1 and λ_2 are imaginary numbers?

- Here the solution trajectories will all be either circles or ellipses with their centres at the origin

- You can determine the direction (clockwise or anticlockwise) and the shape (circular or elliptical) of the trajectories by considering the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for points on the coordinate axes
  - For example, consider the system $\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\boldsymbol{x}$
    - The eigenvalues of $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$ are i and $-\mathrm{i}$, so the trajectories will be elliptical or circular
  - When $x = 1$ and $y = 0$, $\frac{dx}{dt} = 1(1) - 2(0) = 1$ and $\frac{dy}{dt} = 1(1) - 1(0) = 1$

    - This shows that from a point on the positive x-axis the solution trajectory will be moving 'to the right and up' in the direction of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- When $x = 0$ and $y = 1$, $\frac{dx}{dt} = 1(0) - 2(1) = -2$ and $\frac{dy}{dt} = 1(0) - 1(1) = -1$
    - This shows that from a point on the positive y-axis the solution trajectory will be moving 'to the left and down' in the direction of the vector $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$
- The directions of the trajectories at those points tell us that the directions of the trajectories will be anticlockwise
- They also tell us that the trajectories will be ellipses
    - For circular trajectories, the direction of the trajectories when they cross a coordinate axis will be perpendicular to that coordinate axis

What does the phase portrait look like when λ_1 and λ_2 are complex numbers?

- In this case λ_1 and λ_2 will be complex conjugates of the form $a \pm b\mathrm{i}$, where a and b are non-zero real numbers
    - If $a = 0$, $b \neq 0$, then we have the imaginary eigenvalues case above
- Here the solution trajectories will all be spirals
    - If the real part of the eigenvalues is **positive** (i.e., if $a > 0$), then the trajectories will spiral **away from** the origin
    - If the real part of the eigenvalues is **negative** (i.e., if $a < 0$), then the trajectories will spiral **towards** the origin

- You can determine the direction (clockwise or anticlockwise) of the trajectories by considering the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ for points on the coordinate axes
  - For example, consider the system $\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}\boldsymbol{x}$
    - The eigenvalues of $\begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix}$ are $2+3\mathrm{i}$ and $2-3\mathrm{i}$, so the trajectories will be spirals
    - Because the real part of the eigenvalues (2) is positive, the trajectories will spiral away from the origin
  - When $x=1$ and $y=0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 1(1)+5(0) = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = -2(1)+3(0) = -2$
    - This shows that from a point on the positive x-axis the solution trajectory will be moving 'to the right and down' in the direction of the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
- When $x=0$ and $y=1$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 1(0)+5(1) = 5$ and $\frac{\mathrm{d}t}{\mathrm{d}t} = -2(0)+3(1) = 3$
  - This shows that from a point on the positive y-axis the solution trajectory will be moving 'to the right and up' in the direction of the vector $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$
- The directions of the trajectories at those points tell us that the directions of the trajectory spirals will be clockwise

Worked example

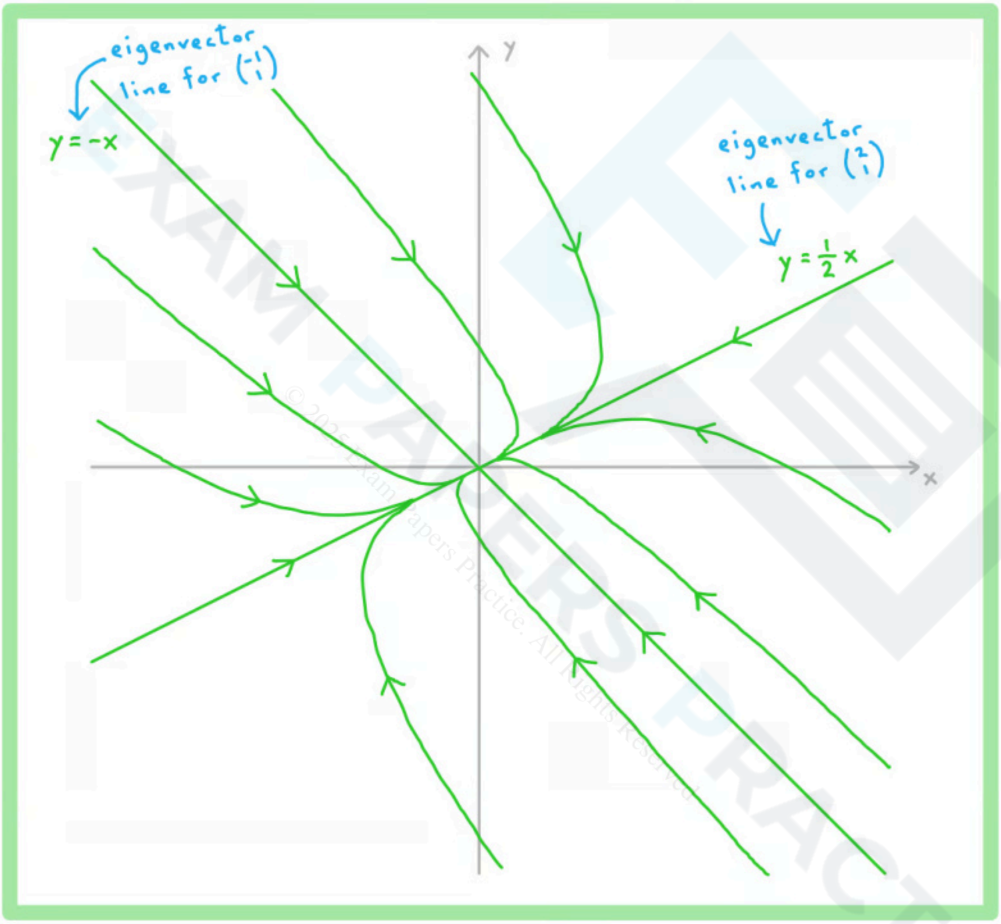
Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2x + 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 3y$$

Given that -1 and -4 are the eigenvalues of the matrix $\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$, with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, draw a phase portrait for the solutions of the system.

Both eigenvalues are negative, so all trajectories will converge on the origin. -4 is more negative than -1, so away from the origin the trajectories will curve towards the $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector line.



Equilibrium Points

What is an equilibrium point?

- For a system of coupled differential equations, an **equilibrium point** is a point (x, y) at which both $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$
  - Because both derivatives are zero, the rates of change of both x and y are zero
  - This means that x and y will not change, and therefore that if the system is ever at the point (x, y) then it will remain at that point (x, y) forever
- An equilibrium point can be **stable** or **unstable**
  - An equilibrium point is stable if for **all** points close to the equilibrium point the solution trajectories move back towards the equilibrium point
    - This means that if the system is perturbed away from the equilibrium point, it will tend to move back towards the state of equilibrium
  - If an equilibrium point is not stable, then it is unstable
    - If a system is perturbed away from an unstable equilibrium point, it will tend to continue moving further and further away from the state of equilibrium

- For a system that can be represented in the matrix form $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}$, where $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, the origin $(0, 0)$ is always an equilibrium point
  - Considering the nature of the **phase portrait** for a particular system will tell us what sort of equilibrium point the origin is
  - If both eigenvalues of the matrix $\boldsymbol{M}$ are **real and negative**, then the origin is a **stable** equilibrium point
    - This sort of equilibrium point is sometimes known as a **sink**
  - If both eigenvalues of the matrix $\boldsymbol{M}$ are **real and positive**, then the origin is an **unstable** equilibrium point
    - This sort of equilibrium point is sometimes known as a **source**
  - If both eigenvalues of the matrix $\boldsymbol{M}$ are **real**, with **one positive and one negative**, then the origin is an **unstable** equilibrium point
    - This sort of equilibrium point is known as a **saddle point** (you will be expected to identify saddle points if they occur in an AI HL exam question)
  - If both eigenvalues of the matrix $\boldsymbol{M}$ are **imaginary**, then the origin is an **unstable** equilibrium point
    - Recall that for all points other than the origin, the solution trajectories here all 'orbit' around the origin along circular or elliptical paths
  - If both eigenvalues of the matrix $\boldsymbol{M}$ are **complex with a negative real part**, then the origin is an **stable** equilibrium point
    - All solution trajectories here spiral in towards the origin

- If both eigenvalues of the matrix M are **complex with a positive real part**, then the origin is an **unstable** equilibrium point
    - All solution trajectories here spiral away from the origin

Worked example

a) Consider the system of coupled differential equations

$$\frac{dx}{dt} = 2x - 3y + 6$$

$$\frac{dy}{dt} = x + y - 7$$

Show that $(3, 4)$ is an equilibrium point for the system.

When $x = 3$ and $y = 4$,

$$\frac{dx}{dt} = 2(3) - 3(4) + 6 = 0$$

$$\frac{dy}{dt} = 3 + 4 - 7 = 0$$

$\frac{dx}{dt}$ and $\frac{dy}{dt}$ are both zero at $(3, 4)$, therefore $(3, 4)$ is an equilibrium point for the system.

b) Consider the system of coupled differential equations

$$\frac{dx}{dt} = x + 3y$$

$$\frac{dy}{dt} = 2x + 2y$$

Given that 4 and -1 are the eigenvalues of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, determine the coordinates and nature of the equilibrium point for the system.

When $x = 0$ and $y = 0$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

Therefore the origin $(0, 0)$ is the equilibrium point for the system.

One of eigenvalues is positive and one is negative.

Therefore the origin is a saddle point, which is an unstable equilibrium point.

Sketching Solution Trajectories

How do I sketch a solution trajectory for a system of coupled differential equations?

- A **phase portrait** shows typical trajectories representing all the possible solutions to a system of coupled differential equations
- For a given set of initial conditions, however, the solution will only have one specific trajectory
- **Sketching a particular solution trajectory** will generally involve the following:
  - Make sure you know what the 'typical' solutions for the system look like
    - You don't need to sketch a complete phase portrait unless asked, but you should know what the phase portrait for your system would look like
    - If the phase portrait includes 'eigenvector lines', however, it is worth including these in your sketch to serve as guidelines
  - Mark the starting point for your solution trajectory
    - The coordinates of the starting point will be the x and y values when $t = 0$
    - Usually these are given in the question as the initial conditions for the system
  - Determine the initial direction of the solution trajectory
    - To do this find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $t = 0$
    - This will tell you the directions in which x and y are changing initially
    - For example if $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 3$ when $t = 0$, then the trajectory from the starting point will initially be 'to the left and up', parallel to the vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
  - Use the above considerations to create your sketch
    - The trajectory should begin at the starting point (be sure to mark and label the starting point on your sketch!)
    - It should move away from the starting point in the correct initial direction
    - As it moves further away from the starting point, the trajectory should conform to the nature of a 'typical solution' for the system

Worked example

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 5y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3x + 3y$$

The initial conditions of the system are such that the exact solution is given by

$$\boldsymbol{x} = \mathrm{e}^{6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - 2\mathrm{e}^{-2t}\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

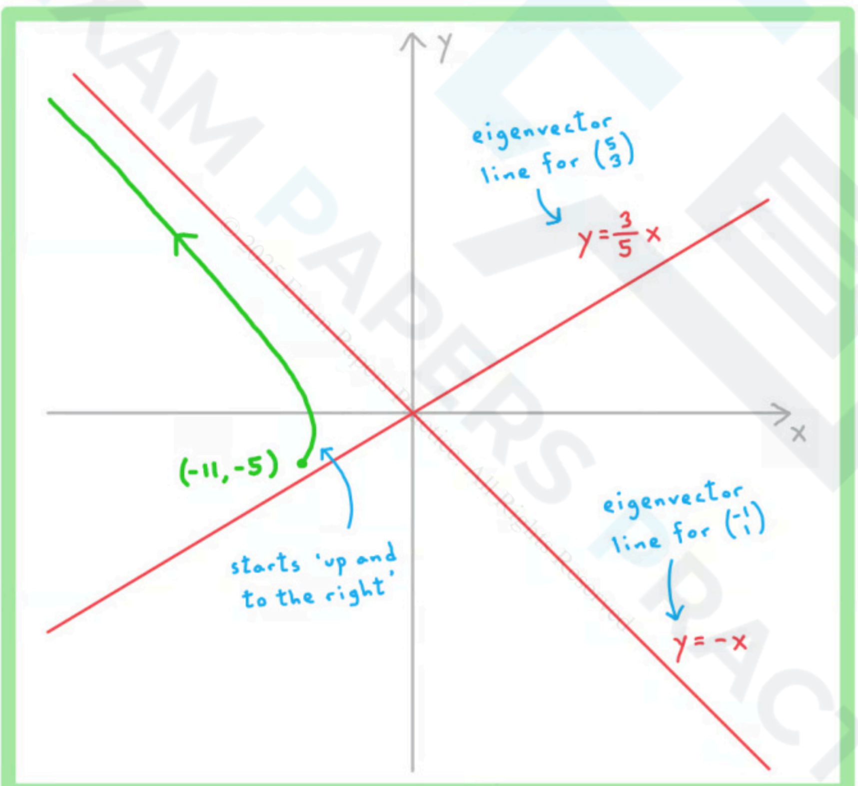
Sketch the trajectory of the solution, showing the relationship between x and y as t increases from zero.

The eigenvalues have different signs, so the trajectory will become approximately parallel to the positive eigenvalue's eigenvector line as it moves away from the origin.

When $t = 0$, $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix}$ starting point

$$\frac{dx}{dt} = (-11) - 5(-5) = 14 \qquad \frac{dy}{dt} = -3(-11) + 3(-5) = 18$$

So the initial trajectory will be 'up and to the right' in the direction of the vector $\begin{pmatrix} 14 \\ 18 \end{pmatrix}$.



5.7.2 Second Order Differential Equations

Euler's Method: Second Order

How do I apply Euler's method to second order differential equations?

- A **second order differential equation** is a differential equation containing one or more second derivatives
- In this section of the course we consider second order differential equations of the form

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$$

  - You may need to rearrange the differential equation given to get it in this form
- In order to apply Euler's method, use the substitution $y = \frac{dx}{dt}$ to turn the second order differential equation into a pair of coupled first order differential equations
  - If $y = \frac{dx}{dt}$, then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$
  - This changes the second order differential equation into the coupled system
- Approximate solutions to this coupled system can then be found using the standard Euler's method for coupled systems
  - See the notes on this method in the revision note 5.6.4 Approximate Solutions to Differential Equations

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = f(x, y, t)$$

Worked example

Consider the second order differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 50\cos t$.

a) Show that the equation above can be rewritten as a system of coupled first order differential equations.

$$\frac{d^2x}{dt^2} = -x - 2\frac{dx}{dt} + 50\cos t \qquad \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$$

Let $y = \frac{dx}{dt}$. Substitution

Then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, so the equation becomes

$$\frac{dy}{dt} = -x - 2y + 50\cos t$$

This gives the coupled system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x - 2y + 50\cos t$$

b)

Initially $x = 2$ and $\frac{dx}{dt} = -1$. By applying Euler's method with a step size of 0.1, find approximations for the values of x and $\frac{dx}{dt}$ when $t = 0.5$.

Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

From exam formula booklet

$x_{n+1} = x_n + 0.1y_n \qquad t_{n+1} = t_n + 0.1$

$y_{n+1} = y_n + 0.1(-x_n - 2y_n + 50\cos t_n)$

Initially $x = 2$ and $y = \frac{dx}{dt} = -1$

n	t_n	x_n	y_n
0	0	2	-1
1	0.1	1.9	4
2	0.2	2.3	7.985
3	0.3	3.0985	11.058
4	0.4	4.2043	13.313
5	0.5	5.5356	14.835

from GDC

At $t = 0.5$,

$x = 5.54$ (3 s.f.)

$\frac{dx}{dt} = 14.8$ (3 s.f.)

Exact Solutions & Phase Portraits: Second Order

How can I find the exact solution for a second order differential equation?

- In some cases we can apply methods we already know to find the exact solutions for second order differential equations
- In this section of the course we consider second order differential equations of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+a\frac{\mathrm{d}x}{\mathrm{d}t}+bx=0$$

  - are constants
- Use the substitution $y=\frac{\mathrm{d}x}{\mathrm{d}t}$ to turn the second order differential equation into a pair of coupled first order differential equations
  - If $y=\frac{\mathrm{d}x}{\mathrm{d}t}$, then $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$
  - This changes the second order differential equation into the coupled system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t}=-bx-ay$$

  - The coupled system may also be represented in matrix form as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -b & -a\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

    - In the 'dot notation' here and
  - That can be written even more succinctly as $\dot{\boldsymbol{x}}=\boldsymbol{M}\boldsymbol{x}$
    - Here $\dot{\boldsymbol{x}}=\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$, $\boldsymbol{x}=\begin{pmatrix}x\\ y\end{pmatrix}$, and $\boldsymbol{M}=\begin{pmatrix}0 & 1\\ -b & -a\end{pmatrix}$
- Once the original equation has been rewritten in matrix form, the standard method for finding exact solutions of systems of coupled differential equations may be used
  - The solutions will depend on the eigenvalues and eigenvectors of the matrix $\boldsymbol{M}$
  - For the details of the solution method see the revision note 5.7.1 Coupled Differential Equations
  - Remember that exam questions will only ask for exact solutions for cases where the eigenvalues of $\boldsymbol{M}$ are real and distinct

How can I use phase portraits to investigate the solutions to second order differential equations?

- Here we are again considering second order differential equations of the form

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

- *a* & *b* are real constants
- As shown above, the substitution $y = \frac{dx}{dt}$ can be used to convert this second order differential equation into a system of coupled first order differential equations of the form $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}$
  - Here $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$
- Once the equation has been rewritten in this form, you may use the standard methods to construct a phase portrait or sketch a solution trajectory for the equation
  - For the details of the phase portrait and solution trajectory methods see the revision note 5.7.1 Coupled Differential Equations
  - When interpreting a phase portrait or solution trajectory sketch, don't forget that $y = \frac{dx}{dt}$
    - So if *x* represents the displacement of a particle, for example, then $y = \frac{dx}{dt}$ will represent the particle's velocity

Worked example

Consider the second order differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} - 4x = 0$. Initially $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$.

a) Show that the equation above can be rewritten as a system of coupled first order differential equations.

$$\frac{d^2x}{dt^2} = 4x - 3\frac{dx}{dt} \qquad \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)$$

Let $y = \frac{dx}{dt}$. Substitution

Then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, so the equation becomes

$$\frac{dy}{dt} = 4x - 3y$$

This gives the coupled system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = 4x - 3y$$

b)

Given that the matrix $\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$ has eigenvalues of 1 and -4 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$, find the exact solution to the second order differential equation.

Exact solution for coupled linear differential equations	$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$

From exam formula booklet

We have $\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}\underline{x}$, so

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-4t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

At $t = 0$, $x = 3$ and $y = \frac{dx}{dt} = -2$, so

$$\begin{pmatrix} A - B \\ A + 4B \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \Rightarrow A = 2, \ B = -1$$

$$\Rightarrow \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = 2e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-4t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$x = 2e^{t} + e^{-4t}$$

c) Sketch the trajectory of the solution to the equation on a phase diagram, showing the relationship between x and $\frac{dx}{dt}$.

At $t = 0$, $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 4(3) - 3(-2) = 18$.

So initially the solution trajectory is to the left and up.