5.5 Optimisation

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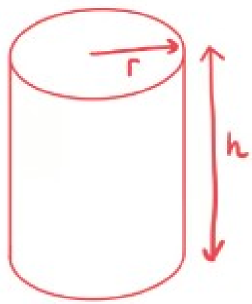
Course	DP IB Maths
Section	5. Calculus
Topic	5.5 Optimisation
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1

a)



The surface area for a cylinder is:

$$S = 2\pi r^2 + h2\pi r$$

Find h in terms of r to eliminate h.

$$Vol = 150\pi = \pi r^2 h$$

$$h = \frac{150}{r^2}$$

$$S = 2\pi r^2 + \frac{150(2\pi \cancel{r})}{r^{\cancel{2}}}$$

$$\boxed{S = 2\pi r^2 + \frac{300\pi}{r}}$$

b) $S = 2\pi r^2 + 300\pi r^{-1}$

Minimum point occurs when gradient, $\left(\frac{dS}{dr}\right) = 0$.

Find r at that minimum.

$$\frac{dS}{dr} = 4\pi r - 300\pi r^{-2} = 0$$

$$4r = \frac{300}{r^2}$$

$$r = \sqrt[3]{75}$$

Sub $r = \sqrt[3]{75}$ into eqn to find minimum value of S.

$$S = 2\pi\left(\sqrt[3]{75}\right)^2 + \frac{300\pi}{\left(\sqrt[3]{75}\right)} = \boxed{335.23\text{cm}^2}$$

(2dp)

Question 2

a) Pythagoras

$r^2 + h^2 = 15^2$

$r^2 + h^2 = 225$

$r^2 = 225 - h^2$

$\boxed{r = \sqrt{225 - h^2}}$

b) Volume of a cone

$V = \frac{1}{3}\pi r^2 h$ (in formula booklet)

$V = \frac{1}{3}\pi\left(\sqrt{225 - h^2}\right)^2 h$

$V = \frac{\pi}{3}(225 - h^2)h$

$\boxed{V = \frac{\pi}{3}(225h - h^3)}$

c) Graph V on your GDC and find its maximum.

$h = 8.66025...$

$\boxed{h = 8.66 \text{ cm} \quad (3sf)}$

Question 3

a) if square perimeter = x

$\therefore$ square sides = $\frac{x}{4}$

$\therefore$ square area = $\left(\frac{x}{4}\right)^2$

$$\boxed{\therefore \text{ square area} = \frac{x^2}{16}}$$

b) $1 \text{ m} = 100 \text{ cm}$

Rectangle perimeter = $2l + 2w$

Rectangle perimeter = $100 - x$

$100 - x = 2l + 2w$ $\quad (l = 2w)$

$100 - x = 2(2w) + 2w$

$100 - x = 6w$

$$\boxed{w = \frac{100 - x}{6}}$$

c) S = rectangle area + square area

$S = lw + \frac{x^2}{16}$ $(l = 2w)$

$S = 2w^2 + \frac{x^2}{16}$

$S = 2\left(\frac{100 - x}{6}\right)^2 + \frac{x^2}{16}$

$S = \cancel{2}\,\frac{(100 - x)^2}{\cancel{36}\ 18} + \frac{x^2}{16}$

$$\boxed{S = \frac{(100 - x)^2}{18} + \frac{x^2}{16}}$$

d) Graph S and find its minimum.

$x = 47.05882\ldots$

$$\boxed{x = 47.1 \text{ cm } (3sf)}$$

Question 4

a) Speed = $\frac{\text{dist } (d)}{\text{time } (t)}$ (not in formula booklet)

$8 = \frac{x}{t_r}$

$$\boxed{t_r = \frac{x}{8}}$$

b) Pythagoras

$BC^2 = 100^2 + (500-x)^2$

$BC^2 = 10\,000 + (500-x)^2$

$$\boxed{BC = \sqrt{10\,000 + (500-x)^2}}$$

c) Speed $= \dfrac{\text{dist } (d)}{\text{time } (t)}$ (not in formula booklet)

Let T be the total time, t_r be the time running and t_s be the time swimming.

$$2 = \frac{\sqrt{10\,000 + (500-x)^2}}{t_s}$$

$$t_s = \frac{\sqrt{10\,000 + (500-x)^2}}{2}$$

$T = t_r + t_s$

$$\boxed{T = \frac{x}{8} + \frac{\sqrt{10\,000 + (500-x)^2}}{2}}$$

d) Graph T and find its minimum.

$x = 474.1801\ldots$

$$\boxed{x = 474 \text{ m} \quad (3\text{sf})}$$

Question 5

a) Volume of a cylinder

$V = \pi r^2 h$ (in formula booklet)

$1000 = \pi r^2 h$

$$\boxed{h = \frac{1000}{\pi r^2}}$$

b) Top skin = cost × circle area

Top skin $= 25\pi r^2$

Curved surface area of a cone

$A = 2\pi r h$ (in formula booklet)

Curved surface = cost × area

Curved surface $= 20 \times 2\pi r h$ $\left(h = \frac{1000}{\pi r^2}\right)$

Curved surface $= 20 \times 2\cancel{\pi}\cancel{r}\left(\frac{1000}{\cancel{\pi} r^{\cancel{2}}}\right)$

Curved surface $= \frac{40\,000}{r}$

$$\boxed{\therefore\ C = 25\pi r^2 + \frac{40\,000}{r}}$$

Question 1

c) Derivative of x^n formula (in formula booklet)

$$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$$

$$C = 25\pi r^2 + 40\,000 r^{-1}$$

$$\frac{dC}{dr} = 50\pi r - 40\,000 r^{-2}$$

$$\boxed{\frac{dC}{dr} = 50\pi r - \frac{40\,000}{r^2}}$$

d) Graph C and find its minimum

$r = p = 6.3384\ldots$

$\boxed{p = 6.34 \text{ cm (3sf)}}$

e) Use your graph from part (d).

$C_{min} = 9466.1027\ldots$ cents

$\boxed{C_{min} = \$95 \text{ (nearest dollar)}}$

f) First derivative = gradient

Second derivative : concavity

Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$\frac{dC}{dr} = 50\pi r - 40\,000r^{-2}$

$\frac{d^2C}{dr^2} = 50\pi + 80\,000r^{-3}$

$\boxed{\frac{d^2C}{dr^2} = 50\pi + \frac{80\,000}{r^3}}$

$50\pi + \frac{80000}{r^3} > 0$ when $x = 6.34$

$\boxed{\therefore C \text{ is concave up at } x = p.}$

Question 6 a) $x = 0$ when no shoes are produced.

$C(0) = 1225 + 11(0) - 0.009(0)^2 - 0.0001(0)^3$

$\boxed{C(0) = 1225 \text{ USD}}$

b) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$C(x) = 1225 + 11x - 0.009x^2 - 0.0001x^3$

Apply formula

$\boxed{C'(x) = 11 - 0.018x - 0.0003x^2}$

c) i) Sub $x = 40$ into $C'(x)$

$C'(40) = 11 - 0.018(40) - 0.0003(40)^2$

$\boxed{C'(40) = 9.80 \text{ USD}}$

ii) Sub $x = 90$ into $C'(x)$

$C'(90) = 11 - 0.018(90) - 0.0003(90)^2$

$\boxed{C'(90) = 6.95 \text{ USD}}$

d) Optimum level of production is when

$R'(x) = C'(x)$

$4.5 = 11 - 0.018x - 0.0003x^2$

Solve for x on your GDC

$x = 120.222\ldots$

120 pairs of running shoes.

Question 7

a) Sub $t = 60$ into $h(t)$.

$$h(60) = -\frac{1}{24}(60)^2 + 3(60) + 12$$

$$\boxed{h(60) = 42\text{ m}}$$

b) Derivative of x^n formula (in formula booklet)

$$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$$

$$h(t) = -\frac{1}{24}t^2 + 3t + 12$$

$$\boxed{h'(t) = -\frac{1}{12}t + 3}$$

c) Set $h'(t) = 0$ and solve for t.

$$-\frac{1}{12}t + 3 = 0$$

$$\boxed{t = 36\text{ s}}$$

Sub $t = 36$ into $h(t)$.

$$h(36) = -\frac{1}{24}(36)^2 + 3(36) + 12$$

$$\boxed{h(36) = 66\text{m}}$$

Question 8

a) Sub $x = 0$ into $C(x)$.

$C(0) = 6(0)^3 - 10(0)^2 + 10(0) + 4$

$C(0) = 4$

$\boxed{400 \text{ AUD}}$

b) $P(x) = R(x) - C(x)$

$P(x) = 42x - (6x^3 - 10x^2 + 10x + 4)$

$\boxed{P(x) = -6x^3 + 10x^2 + 32x - 4}$

c) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$P(x) = -6x^3 + 10x^2 + 32x - 4$

$\boxed{P'(x) = -18x^2 + 20x + 32}$

d) Set $P'(x) = 0$ and solve for x.

$-18x^2 + 20x + 32 = 0$

$x = 2$ $\quad$ ~~$x = -0.889$~~

Reject as $P(x) > 0$ and $x \geq 0$

Sub $x = 2$ into $P(x)$.

$P(2) = -6(2)^3 + 10(2)^2 + 32(2) - 4$

$P(2) = 52$

∴ The profit maximising production level is 200 bats and the expected profit is 5200 AUD.

Question 9 a) Volume of a cuboid formula.

$V = lwh$ (in formula booklet)

where l is the length, w is the width and h is the height.

$l = 55 - 2x$ $\quad$ $w = 28 - 2x$ $\quad$ $h = x$

Sub l, w and h into formula.

$V = (55 - 2x)(28 - 2x)x$ expand and rearrange

$$\boxed{V = 4x^3 - 166x^2 + 1540x}$$

b) Derivative of x^n formula (in formula booklet)

$$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$$

$$V = 4x^3 - 166x^2 + 1540x$$

$$\boxed{\frac{dV}{dx} = 12x^2 - 332x + 1540}$$

c)i) Set $\frac{dV}{dx} = 0$ and solve for x.

$$12x^2 - 332x + 1540 = 0$$

$x = 5.8943...$ $\quad \cancel{x = 21.772...}$

Reject as $l > 0$

$$\boxed{x = 5.89\text{cm} \quad (3sf)}$$

ii) Sub $x = 5.8943...$ into V.

$$V = 4(5.8943)^3 - 166(5.8943)^2 + 1540(5.8943)$$

$$V = 4129.059...$$
$$= 4130 \quad (3sf)$$

$$\boxed{V = 4.13 \times 10^3 \text{ cm}^3}$$