DP IB Maths: AA HL

5.5 Optimisation

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5.5.1 Modelling with Differentiation

Modelling with Differentiation

What can be modelled with differentiation?

- Recall that **differentiation** is about the **rate** of **change** of a function and provides a way of finding **minimum** and **maximum** values of a function
- Anything that involves **maximising** or **minimising** a quantity can be modelled using differentiation; for example
  - minimising the cost of raw materials in manufacturing a product
  - the maximum height a football could reach when kicked
- These are called **optimisation** problems

What modelling assumptions are used in optimisation problems?

- The quantity being optimised needs to be dependent on a **single** variable
  - If other variables are initially involved, **constraints** or **assumptions** about them will need to be made; for example
    - minimising the cost of the **main** raw material – timber in manufacturing furniture say
    - the cost of screws, glue, varnish, etc can be fixed or considered **negligible**
- Other **modelling assumptions** may have to be made too; for example
  - ignoring air resistance and wind when modelling the path of a kicked football

How do I solve optimisation problems?

- In **optimisation** problems, letters other than ***x***, ***y*** and ***f*** are often used including capital letters
  - ***V*** is often used for volume, ***S*** for surface area
  - ***r*** for radius if a circle, cylinder or sphere is involved
- **Derivatives** can still be found but be clear about which letter is representing the independent (*x*) variable and which letter is representing the dependent (*y*) variable
  - A GDC may always use *x* and *y* but ensure you use the correct letters throughout your working and final answer
- Problems often start by **linking** two connected quantities together – for example **volume** and **surface area**
  - Where more than one variable is involved, **constraints** will be given such that the quantity of interest can be rewritten in terms of **one** variable
- Once the quantity of interest is written as a function of a single variable, **differentiation** can be used to **maximise** or **minimise** the quantity as required

STEP 1
Rewrite the quantity to be optimised in terms of a single variable, using any constraints given in the question

STEP 2

Differentiate and solve the derivative equal to zero to find the "x"-coordinate(s) of any stationary points

STEP 3
If there is more than one stationary point, or the requirement to justify the nature of the stationary point, differentiate again

STEP 4
Use the second derivative to determine the nature of each stationary point and select the maximum or minimum point as necessary

STEP 5
Interpret the answer in the context of the question

Worked example

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.

The total area of the bed is to be $100\pi \text{ m}^2$.

a) Show that the perimeter of the bed is given by the formula

$$P = \pi\left(r + \frac{100}{r}\right)$$

STEP 1: The width of the rectangle is $2r$ m and its length L m
The AREA of the bed, 100π m^2 is given by

$$\underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} + \underbrace{2rL}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} = \underbrace{100\pi}_{\text{total area (constraint)}}$$

$$\therefore \pi r^2 + 2rL = 100\pi$$
$$2rL = 100\pi - \pi r^2$$
$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

Write L in terms of r

The PERIMETER of the bed is

$$P = \underbrace{\pi r + \pi r}_{\text{semi-circular arcs}} + \underbrace{2L}_{\text{two straight lengths}}$$

Use L from the area constraint to write P in terms of r only

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$
$$P = \pi r + \frac{100\pi}{r}$$

$$\boxed{\therefore P = \pi\left(r + \frac{100}{r}\right)}$$

b) Find $\dfrac{\mathrm{d}P}{\mathrm{d}r}$.

STEP 1: Rewrite P as powers of r

$$P = \pi\left(r + 100r^{-1}\right)$$

STEP 2: $\dfrac{dP}{dr} = \pi\left(1 - 100r^{-2}\right)$

$$\boxed{\therefore \frac{dP}{dr} = \pi\left(1 - \frac{100}{r^2}\right)}$$

c) Find the value of r that minimises the perimeter.

STEP 2: $\pi\left(1 - \dfrac{100}{r^2}\right) = 0$

$r^2 - 100 = 0$

$r = 10$ (reject -10 as r is a length)

This is the only stationary point so we can assume it is minimal.

$$\boxed{\therefore r = 10 \text{ m minimises the perimeter}}$$

d) Hence find the minimum perimeter.

STEP 5: Interpret answer in context

Minimum perimeter is when $r=10$

$\therefore P = \pi\left(10 + \frac{100}{10}\right) = 20\pi$

$\boxed{\text{Minimum perimeter is } 20\pi \text{ m}}$

Use your GDC to check

e) Justify that this is the minimum perimeter.

STEP 4: Use second derivative

$$\frac{d^2P}{dr^2} = \pi\left(200r^{-3}\right)$$

at $r=10$, $\frac{d^2P}{dr^2} = \frac{\pi}{5} > 0 \quad \therefore$ minimum

$\boxed{\therefore 20\pi \text{ is the minimum perimeter}}$