Exam Papers Practice

5.4 Further Integration

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Course	DP IB Maths
Section	5. Calculus
Topic	5.4 Further Integration
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1

$\int \sin x \, dx = -\cos x + c$ } standard integral

a) $\boxed{-\cos x + c}$

$\int \frac{1}{x} \, dx = \ln|x| + c$ } standard integral

b) $\int_1^4 \frac{1}{x} \, dx = \left[\ln|x|\right]_1^4$

$= \ln|4| - \ln|1|$

$= \ln 4 - \cancel{\ln 1}$ $\quad \ln 1 = 0$

$\boxed{= \ln 4}$

$f(x) = e^x \implies f'(x) = e^x$ } Derivative of e^x

$y = g(u)$ where $u = f(x)$

$\implies \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ } Chain rule

c) Let $y = e^{7x}$. Then $y = e^u$ and $u = 7x$, so

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (e^u)(7) = 7e^u = 7e^{7x}$

And $\int \frac{dy}{dx} \, dx = y + c$ (indefinite integral as antiderivative), so:

$\boxed{\int 7e^{7x} \, dx = e^{7x} + c}$

Question 2

$f(x) = \sin x \implies f'(x) = \cos x$ } Derivative of $\sin x$

$y = g(u)$ where $u = f(x)$

$\implies \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ } Chain rule

a) By the chain rule, $\dfrac{d}{dx}(\sin 2x) = 2\cos 2x$

$$\int \cos 2x \, dx = \underbrace{\tfrac{1}{2}}_{\text{compensate}} \int \overbrace{2}^{\text{adjust}} \cos 2x \, dx$$

$$\boxed{= \tfrac{1}{2} \sin 2x + c}$$

(indefinite integral as antiderivative)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad \Big\} \text{ Integral of } x^n \ (n \neq -1)$$

Substitution

b) Let $u = 3x - 1$. Then:

$$\frac{du}{dx} = 3 \implies du = 3\,dx \implies dx = \frac{1}{3}du \quad \text{Find du in terms of x}$$

$$\left.\begin{array}{l} x = 2 \implies u = 3(2) - 1 = 5 \\ x = 0 \implies u = 3(0) - 1 = -1 \end{array}\right\} \text{Transform integration limits}$$

$$\int_0^2 (3x-1)^3\,dx = \int_{-1}^5 u^3 \left(\frac{1}{3}du\right)$$

$$= \frac{1}{3}\int_{-1}^5 u^3\,du$$

$$= \frac{1}{3}\left[\frac{1}{4}u^4\right]_{-1}^5$$

$$= \frac{1}{3}\left(\frac{1}{4}(5)^4 - \frac{1}{4}(-1)^4\right)$$

$$= \frac{1}{3}\left(\frac{625}{4} - \frac{1}{4}\right) = \frac{624}{12} = \boxed{52}$$

$$f(x) = e^x \implies f'(x) = e^x \quad \Big\} \text{ Derivative of } e^x$$

$$\left.\begin{array}{l} y = g(u) \text{ where } u = f(x) \\ \implies \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} \end{array}\right\} \text{Chain rule}$$

c) By the chain rule, $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

$$y = \int e^{5x}\,dx = \underbrace{\tfrac{1}{5}}_{\text{compensate}} \int \overbrace{5}^{\text{adjust}} e^{5x}\,dx$$

$$\boxed{y = \frac{1}{5}e^{5x} + c}$$

(indefinite integral as antiderivative)

Question 3

$\int \frac{1}{x}\,dx = \ln|x| + c$ } Standard integral

Substitution

Let $u = x + 4$. Then:

$\frac{du}{dx} = 1 \Rightarrow du = dx$ Find du in terms of x

$x = 2 \Rightarrow u = 2 + 4 = 6$
$x = 1 \Rightarrow u = 1 + 4 = 5$ } Transform integration limits

$u = x + 4 \Rightarrow x = u - 4$

$$\int_1^2 \frac{x}{x+4}\,dx = \int_5^6 \frac{u-4}{u}\,du = \int_5^6 \left(1 - \frac{4}{u}\right) du$$

$$= \left[u - 4\ln|u|\right]_5^6$$

$$= (6 - 4\ln 6) - (5 - 4\ln 5)$$

$$= 1 + 4(\ln 5 - \ln 6)$$

$$= 1 + 4\ln\frac{5}{6}$$

$\log_a \frac{x}{y} = \log_a x - \log_a y$ } Exponents and logarithms

Question 4

$$f(x) = \sin x \implies f'(x) = \cos x \quad \Big\} \text{ Derivative of } \sin x$$

$$\left.\begin{array}{l} y = g(u) \text{ where } u = f(x) \\ \implies \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} \end{array}\right\} \text{Chain rule}$$

$$\cos 2\theta = 2\cos^2\theta - 1 \implies \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\int_{\pi/4}^{\pi/2} \cos^2\theta \, d\theta = \int_{\pi/4}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

By the chain rule, $\frac{d}{d\theta}(\sin 2\theta) = 2\cos 2\theta$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{1}{2} + \underbrace{\frac{1}{4}}_{\text{compensate}}(\overbrace{2}^{\text{adjust}}\cos 2\theta)\right) d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{8} + \frac{1}{4}\sin\pi - \frac{1}{4}\sin\frac{\pi}{2}$$

$$\boxed{= \frac{\pi}{8} - \frac{1}{4}}$$

$\sin\pi = 0$

$\sin\frac{\pi}{2} = 1$

Question 5

$f(x) = x^n \implies f'(x) = nx^{n-1}$ } Derivative of x^n

a) $f'(x) = 2(3x^2) + 4 = \boxed{6x^2 + 4}$

$\int \frac{1}{x}\,dx = \ln|x| + c$ } Standard integral

Substitution

b) Let $u = 2x^3 + 4x$. Then:

$\frac{du}{dx} = 6x^2 + 4$ from part (a)

$\implies du = (6x^2 + 4)\,dx$

$\implies \frac{1}{2}\,du = (3x^2 + 2)\,dx$ Find du in terms of x

$$\int \frac{3x^2 + 2}{2x^3 + 4x}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\right)du$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + c$$

$$\boxed{= \frac{1}{2}\ln|2x^3 + 4x| + c}$$

Question 6

a) $x^2 - 3x + 4 = 4 - x^2 + 2x$

$x^2 + x^2 - 3x - 2x + 4 - 4 = 0$

$2x^2 - 5x = 0$

$x(2x - 5) = 0$

So $x = 0$ or $2x - 5 = 0 \Rightarrow x = \frac{5}{2}$

$$\boxed{x = 0 \text{ and } x = \frac{5}{2}}$$

b) This is the area between two curves between $x = 0$ and $x = \frac{5}{2}$, so:

Area $= \int_0^{5/2} [(\text{top curve}) - (\text{bottom curve})]\, dx$

$= \int_0^{5/2} [(4 - x^2 + 2x) - (x^2 - 3x + 4)]\, dx$

$= \int_0^{5/2} (4 - 4 + 2x + 3x - x^2 - x^2)\, dx$

$= \int_0^{5/2} (5x - 2x^2)\, dx$

c) Area of $R = \int_0^{5/2} (5x - 2x^2)\,dx$

$$\boxed{= \frac{125}{24} \text{ units}^2}$$ from GDC

You could also work this out by hand via

$$\int_0^{5/2} (5x - 2x^2)\,dx = \left[\frac{5}{2}x^2 - \frac{2}{3}x^3\right]_0^{5/2}$$

$$= \left(\frac{125}{8} - \frac{250}{24}\right) - (0 - 0) = \frac{125}{24}$$

Question 7 a) $x^2 = 0 \Rightarrow x = 0$ $\quad y = 0$ on the x-axis

$\boxed{\text{The x-coordinate of } P = 0}$

$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

But $x > 0$, so:

$\boxed{\text{The x-coordinate of } Q = 2}$

$6 - x = 0 \Rightarrow x = 6$ $\quad y = 0$ on the x-axis

$\boxed{\text{The x-coordinate of } R = 6}$

or $y = 6 - 2 = 4$

b) When $x = 2$, $y = (2)^2 = 4 \Rightarrow Q$ is the point $(2, 4)$

$A = \frac{1}{2}(bh)$ } Area of a triangle (b = base, h = height)

$\frac{1}{2}(4 \times 4) = \frac{1}{2}(16) = 8$ area of the triangle

$A = \int_a^b y \, dx$ } Area between a curve $y = f(x)$ and the x-axis, where $f(x) > 0$

$$\int_0^2 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{(2)^3}{3} - \frac{(0)^3}{3} = \frac{8}{3} - 0 = \frac{8}{3}$$

You could also evaluate the integral with your GDC.

Area of $R = 8 + \frac{8}{3} = \frac{32}{3}$

$\boxed{\text{Area of } R = \frac{32}{3} \text{ units}^2}$

Question 8

$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$ [Property of definite integrals]

a) $\int_5^1 h(x) \, dx = -\int_1^5 h(x) \, dx = \boxed{-2}$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \Big\} \text{ Integral of } x^n \ (n \neq -1)$$

$$\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx \qquad \left[\begin{array}{c}\text{Properties of} \\ \text{definite integrals}\end{array}\right]$$

$$\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

b) $\displaystyle\int_1^5 \frac{h(x)+1}{2}\, dx = \int_1^5 \frac{1}{2}\left(h(x)+1\right) dx$

$$= \frac{1}{2}\int_1^5 \left(h(x)+1\right) dx$$

$$= \frac{1}{2}\left(\int_1^5 h(x)\, dx + \int_1^5 1\, dx\right)$$

And $\displaystyle\int_1^5 1\, dx = [x]_1^5 = 5 - 1 = 4$, so

$$\int_1^5 \frac{h(x)+1}{2}\, dx = \frac{1}{2}(2+4) = \boxed{3}$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \Big\} \text{ Integral of } x^n \ (n \neq -1)$$

$$\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

[Property of definite integrals]

c) $\int_1^5 (h(x) + 2x)\, dx = \int_1^5 h(x)\, dx + \int_1^5 2x\, dx$

And $\int_1^5 2x\, dx = [x^2]_1^5 = 5^2 - 1^2 = 25 - 1 = 24$

So $\int_1^5 (h(x) + 2x)\, dx = 2 + 24 = \boxed{26}$

Question 9

$$f(x) = x^n \implies f'(x) = nx^{n-1} \quad \Big\} \text{ Derivative of } x^n$$

$$f(x) = \ln x \implies f'(x) = \frac{1}{x} \quad \Big\} \text{ Derivative of } \ln x$$

$$\left. \begin{array}{l} y = g(u) \text{ where } u = f(x) \\ \implies \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} \end{array} \right\} \text{ Chain rule}$$

a) $f'(x) = \frac{1}{2x^2 + 1}(4x)$

$$\boxed{f'(x) = \frac{4x}{2x^2 + 1}}$$

b) $\int \frac{x}{2x^2+1}\,dx = \underbrace{\frac{1}{4}}_{\text{compensate}} \int \frac{\overbrace{4x}^{\text{adjust}}}{2x^2+1}\,dx$

$$\boxed{= \frac{1}{4}\ln(2x^2+1) + c}$$

$\int \frac{dy}{dx}\,dx = y + c$ (indefinite integral as antiderivative)

Question 10

$$\left.\begin{aligned} &y = g(u) \text{ where } u = f(x) \\ &\Longrightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \end{aligned}\right\} \text{Chain rule}$$

$f(x) = \sin x \Longrightarrow f'(x) = \cos x$ } Derivative of $\sin x$

$$y = \sin(u) \qquad u = x^3 + 1$$

Let $y = \sin\big(\underbrace{x^3 + 1}_{u = x^3+1}\big)$. Then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (\cos(x^3 + 1))(3x^2) = 3x^2 \cos(x^3 + 1)$$

adjust

$$f(x) = \int x^2 \cos(x^3 + 1)\, dx = \frac{1}{3} \int 3x^2 \cos(x^3 + 1)\, dx$$

compensate

$$\int \frac{dy}{dx}\, dx = y + c \quad \left(\begin{matrix}\text{indefinite integral}\\ \text{as antiderivative}\end{matrix}\right)$$

$$\Longrightarrow \quad f(x) = \frac{1}{3} \sin(x^3 + 1) + c$$

But $f(-1) = 1$, so

$$\frac{1}{3} \sin((-1)^3 + 1) + c = 1$$

$$\frac{1}{3} \sin(0) + c = 1 \Longrightarrow c = 1 \qquad \sin(0) = 0$$

$$\boxed{f(x) = \frac{1}{3} \sin(x^3 + 1) + 1}$$