



5.4 Further Integration

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5.4.1 Integrating Special Functions

Integrating Trig Functions

How do l integrate sin, cos and sec^2?

• The antiderivatives for sine and cosine are

 $\int \sin x \, dx = -\cos x + c$ $\int \cos x \, dx = \sin x + c$

where C is the constant of integration

Also, from the **derivative** of tan x

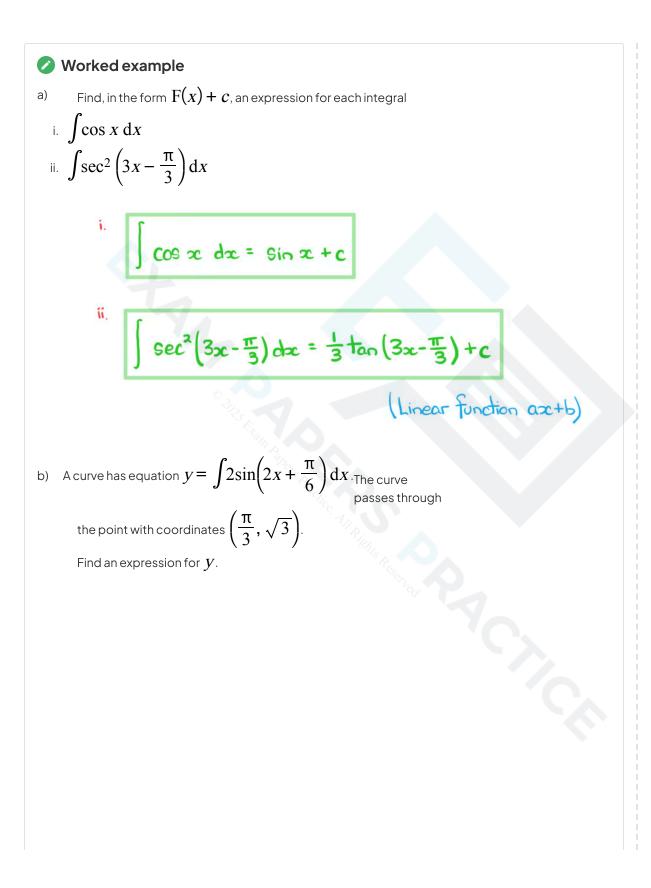
$$\int \sec^2 x \, \mathrm{d}x = \tan x + c$$

- The derivatives of Sin X, COS X and tan X are in the formula booklet
 - so these **antiderivatives** can be easily deduced
- For the linear function ax + b, where a and b are constants,

$$\int \sin (ax+b) dx = -\frac{1}{a} \cos (ax+b) + c$$
$$\int \cos (ax+b) dx = \frac{1}{a} \sin (ax+b) + c$$
$$\int \sec^2 (ax+b) dx = \frac{1}{a} \tan (ax+b) + c$$

- For calculus with trigonometric functions angles must be measured in radians
 - Ensure you know how to change the angle mode on your GDC







$$y = \lambda \int \sin(2x + \frac{\pi}{6}) dx$$

$$y = \lambda \left[-\frac{1}{2}\cos(2x + \frac{\pi}{6}) \right] + c$$

$$At = \frac{\pi}{3}, y = \sqrt{3}, \quad \sqrt{3} = -\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) + c$$

$$c = \cos\left(\frac{5\pi}{6}\right) + \sqrt{3}$$

$$c = \frac{\sqrt{3}}{2}$$

$$\therefore y = \sqrt{3} - \cos\left(2x + \frac{\pi}{6}\right)$$



Integrating e^x & 1/x

How do l integrate exponentials and 1/x?

• The antiderivatives involving e^x and $\ln x$ are

$$\int e^x dx = e^x + c$$
$$\int \frac{1}{x} dx = \ln|x| + c$$

where C is the constant of integration

- These are given in the formula booklet
- For the linear function (ax + b), where a and b are constants,

$$\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a} \ln|ax+b| + c$$

• It follows from the last result that

$$\int \frac{a}{ax+b} \, \mathrm{d}x = \ln|ax+b| + c$$

- which can be deduced using **Reverse Chain Rule**
- With In, it can be useful to write the constant of integration, C, as a logarithm
 - using the laws of logarithms, the answer can be written as a single term

$$\int \frac{1}{x} dx = \ln |x| + \ln k = \ln k |x|$$
 where k is a constant

• This is similar to the special case of **differentiating** $\ln(ax + b)$ when b = 0



Worked example

A curve has the gradient function $f'(x) = \frac{3}{3x+2} + e^{4-x}$.

Given the exact value of f(1) is $\ln 10 - e^3$ find an expression for f(x).

$$f(x) = \int \left(\frac{3}{3x+2} + e^{h-x}\right) dx$$

$$f(x) = 3\int \frac{1}{3x+2} dx + \int e^{h-x} dx$$

$$= 3\left[\frac{1}{3}\ln|3x+2|\right] - e^{h-x} + c$$

$$f(1) = \ln 10 - e^{3}, \quad \ln |3x|+2| - e^{h-1} + c = \ln 10 - e^{3}$$

$$\therefore c = \ln 10 - \ln 5$$

$$c = \ln (\frac{10}{5}) = \ln 2$$

$$\therefore f(x) = \ln |3x+2| - e^{h-x} + \ln 2$$

$$(= \ln 2|3x+2| - e^{h-x})$$



5.4.2 Techniques of Integration

Integrating Composite Functions (ax+b)

What is a composite function?

- A composite function involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being linear i.e. of the form ax + b

How do I integrate linear (ax+b) functions?

- A linear function (of x) is of the form ax + b
- The special cases for trigonometric functions and exponential and logarithm functions are

$$\int \sin(ax+b) \, \mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b) \, \mathrm{d}x = \frac{1}{a}\sin(ax+b) + a$$

$$\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a} \ln|ax+b| + c$$

• There is one more special case

•
$$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$$
 where $n \in \mathbb{Q}, n \neq -1$

- *C*, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
 - However, spotting them can make solutions more efficient



Worked example

Find the following integrals

a)
$$\int 3(7-2x)^{\frac{5}{3}} dx$$

$$I = \int 3(7-2x)^{\frac{5}{3}} dx = 3 \int (-2x+7)^{\frac{5}{3}} dx$$

$$Using \int (ax+b)^{n} dx = \frac{1}{a(n+1)} (ax+b)^{n+1} + c,$$

$$I = 3 \left[\frac{1}{-2x \frac{9}{3}} (-2x+7)^{\frac{9}{3}} \right] + c^{\frac{5}{3}+1}$$

$$\therefore I = -\frac{9}{16} (7-2x)^{\frac{9}{3}} + c$$

b)
$$\int \frac{1}{2} \cos(3x-2) \, \mathrm{d}x$$

 $I = \int \frac{1}{2} \cos (3x-2) dx = \frac{1}{2} \int \cos (3x-2) dx$ Using $\int \cos (ax+b) dx = \frac{1}{a} \sin (ax+b) + c$

 $I = \frac{1}{2} \left[\frac{1}{3} \sin(3\infty - 2) \right] + c$

$$\therefore \mathbf{I} = \frac{1}{6} \sin(3x-2) + c$$

S,C



Reverse Chain Rule

What is reverse chain rule?

- The Chain Rule is a way of differentiating two (or more) functions
- **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
 - spotting that chain rule would be used in the reverse (differentiating) process

How do I know when to use reverse chain rule?

- Reverse chain rule is used when we have the product of a composite function and the derivative of its secondary function
- Integration is trickier than differentiation; many of the shortcuts do not work
 - For example, in general $\int e^{f(x)} dx \neq \frac{1}{f'(x)} e^{f(x)}$
 - However, this result is true if f(x) is linear (ax + b)
- Formally, in function notation, reverse chain rule is used for integrands of the form

$$I = \int g'(x) f'(g(x)) \, \mathrm{d}x$$

- this does not have to be strictly true, but 'algebraically' it should be
 - if coefficients do not match 'adjust and compensate' can be used
 - e.g. $5x^2$ is not quite the derivative of $4x^3$
 - the algebraic part (X^2) is 'correct'
 - but the coefficient 5 is 'wrong'
 - use 'adjust and compensate' to 'correct' it
- A particularly useful instance of reverse chain rule to recognise is

$$I = \int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + c$$

- i.e. the **numerator** is (almost) the **derivative** of the **denominator**
- 'adjust and compensate' may need to be used to deal with any coefficients

• e.g.

$$I = \int \frac{x^2 + 1}{x^3 + 3x} \, dx = \frac{1}{3} \int 3 \frac{x^2 + 1}{x^3 + 3x} \, dx = \frac{1}{3} \int \frac{3x^2 + 3}{x^3 + 3x} \, dx = \frac{1}{3} \ln|x^3 + 3x| + c$$

How do I integrate using reverse chain rule?

- If the product **can** be identified, the **integration** can be done "by **inspection**"
 - there may be some "adjusting and compensating" to do
- Notice a lot of the "adjust and compensate method" happens mentally
 - this is indicated in the steps below by quote marks

STEP 1

Spot the 'main' function



e.g.
$$I = \int x(5x^2 - 2)^6 dx$$

"the main function is $(\dots)^6$ which would come from $(\dots)^7$."

STEP 2

'Adjust' and 'compensate' any coefficients required in the integral e.g. " (...)⁷ would differentiate to 7(...)⁶" "chain rule says multiply by the derivative of $5x^2 - 2$, which is 10x" "there is no '7' or '10' in the integrand so adjust and compensate"

$$I = \frac{1}{7} \times \frac{1}{10} \times \int 7 \times 10 \times x(5x^2 - 2)^6 \, \mathrm{d}x$$

STEP 3 Integrate and simplify

e.g.
$$I = \frac{1}{7} \times \frac{1}{10} \times (5x^2 - 2)^7 + c$$

 $I = \frac{1}{70} (5x^2 - 2)^7 + c$

- Differentiation can be used as a means of checking the final answer
 - After some practice, you may find Step 2 is not needed
 - Do use it on more awkward questions (negatives and fractions!)
- If the product **cannot** easily be identified, use **substitution**





A curve has the gradient function $f'(x) = 5x^2 \sin(2x^3)$. Find an expression for f(x).

$$f(x) = \int 5x^{2} \sin(2x^{3}) dx$$

$$f(x) = 5\int x^{2} \sin(2x^{3}) dx$$
Take 5 out as a factor
This is a product, almost in the form $g'(x) f(g(x))$
STEP 1: Spot the 'main' function
$$\int_{x}^{x} \text{ the main function is } \sin(...) \text{ which would}$$
STEP 2: 'Adjust and compensate' coefficients
$$\int_{x}^{x} \cos(...) \text{ would differentiate to } -\sin(...)^{n}$$

$$f(x) = 5x - x \frac{1}{6}x \int -x 6x x^{2} \sin(2x^{3}) dx$$
STEP 3: Integrate and simplify
$$f(x) = -\frac{5}{6} \cos(2x^{3}) + c$$



Substitution: Reverse Chain Rule

What is integration by substitution?

- When reverse chain rule is difficult to spot or awkward to use then integration by substitution can be used
 - substitution simplifies the integral by defining an alternative variable (usually U) in terms of the original variable (usually X)
 - everything (including "dX" and limits for definite integrals) is then substituted which makes the integration much easier

How do l integrate using substitution?

STEP 1

Identify the substitution to be used - it will be the secondary function in the composite function

So
$$g(x)$$
 in $f(g(x))$ and $u = g(x)$

STEP 2

Differentiate the substitution and rearrange

d*u*

 $\frac{dx}{dx}$ can be treated like a fraction

(i.e. "multiply by dX" to get rid of fractions)

STEP 3

Replace all parts of the integral

All x terms should be replaced with equivalent u terms, including dx

If finding a **definite integral** change the limits from *X*-values to *U*-values too

STEP 4

Integrate and either

substitute X back in

or

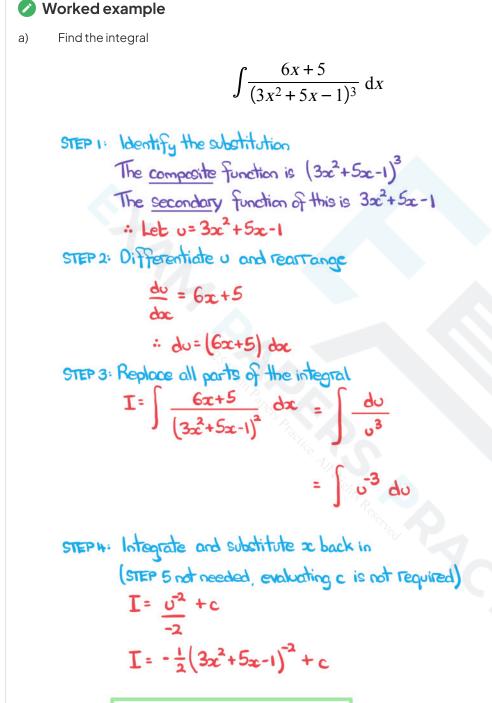
evaluate the definte integral using the *U* limits (either using a GDC or manually)

STEP 5

Find ${\boldsymbol{\mathcal{C}}}$, the constant of integration, if needed

- For definite integrals, a GDC should be able to process the integral without the need for a substitution
 - be clear about whether working is required or not in a question





$$\stackrel{\text{!`I}}{=} \frac{-1}{2(3x^2+5x-1)^2} + c$$



b) Evaluate the integral

$$\int_{1}^{2} \frac{6x+5}{(3x^2+5x-1)^3} \, \mathrm{d}x$$

giving your answer as an exact fraction in its simplest terms.

Note that you could use your GOC for this part Certainly use it to check your answer!

From STEP 3 above, $I = \int_{x=1}^{x=2} u^3 du$ Change limits too, x=1, $u=3(1)^2+5(1)-1=7$ x=2, $u=3(2)^2+5(2)-1=21$

x=2, u=3(2)+5(2)-1=21STEP 4: Integrate and evaluate

$$I = \left[-\frac{1}{2} o^{-2} \right]_{7}^{21} = \left[-\frac{1}{2} (21)^{2} \right] - \left[-\frac{1}{2} (7)^{-2} \right]_{7}^{21}$$



5.4.3 Definite Integrals

Definite Integrals

What is a definite integral?

$$\int_{a}^{b} \mathbf{f}(x) \, dx = \left[\mathbf{F}(x)\right]_{a}^{b} = \mathbf{F}(b) - \mathbf{F}(a)$$

- This is known as the Fundamental Theorem of Calculus
- a and b are called limits
 - a is the lower limit
 - **b** is the upper limit
- f(x) is the integrand
- F(x) is an **antiderivative** of f(x)
- The constant of integration ("+c") is not needed in definite integration
 - "+c" would appear alongside both **F(a)** and **F(b)**
 - subtracting means the "+c"'s cancel

How do I find definite integrals analytically (manually)?

STEP 1

Give the integral a name to save having to rewrite the whole integral every time If need be, rewrite the integral into an integrable form

$$I = \int_{a}^{b} f(x) \, \mathrm{d}x$$

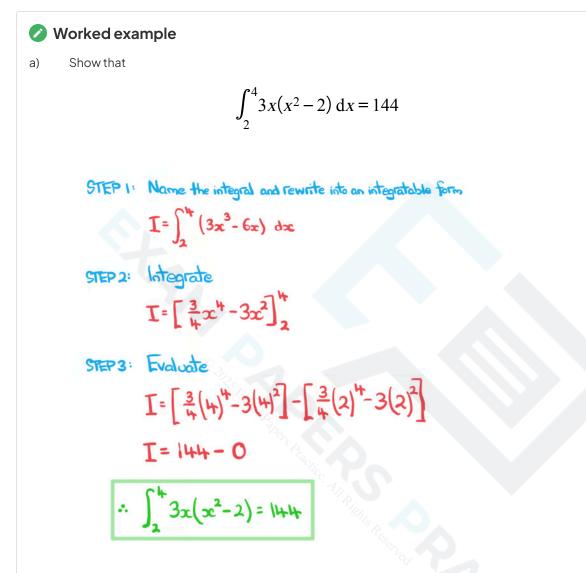
STEP 2

Integrate without applying the limits; you will not need "+c" Notation: use square brackets [] with limits placed at the end bracket

STEP 3

Substitute the limits into the function and evaluate





b) Use your GDC to evaluate

$$\int_0^1 3e^{x^2 \sin x} \, \mathrm{d}x$$

giving your answer to three significant figures.



Using GDC. $\int_{0}^{1} 3e^{x^{2}\sin x} dx = 3.872.957...$

 $: \int_{0}^{1} 3e^{x^{2} \sin x} dx = 3.87 \quad (3 \text{ s.f.})$



Properties of Definite Integrals

Fundamental Theorem of Calculus

$$\int_{a}^{b} \mathbf{f}(x) \, dx = [\mathbf{F}(x)]_{a}^{b} = \mathbf{F}(b) - \mathbf{F}(a)$$

- Formally,
 - f(x) is **continuous** in the interval $a \le x \le b$
 - F(x) is an **antiderivative** of f(x)

What are the properties of definite integrals?

- Some of these have been encountered already and some may seem obvious ...
 - taking constant factors outside the integral

-
$$\int_{a}^{b} kf(x) dx = k \int_{a}^{b} f(x) dx$$
 where k is a constant

- useful when fractional and/or negative values involved
- integrating term by term

$$\int_{a}^{b} [f(x) + g(x)] dx = \int_{a}^{b} f(x) dx + \int_{a}^{b} g(x) dx$$

• the above works for subtraction of terms/functions too

• equal upper and lower limits

$$\int_{a}^{a} f(x) \, dx = 0$$

on evaluating, this would be a value, subtract itself !

• swapping limits gives the same, but negative, result

$$\int_{a}^{b} f(x) dx = - \int_{b}^{a} f(x) dx$$

- compare 8 subtract 5 say, with 5 subtract 8 ...
- splitting the interval

$$\int_{a}^{b} f(x) dx = \int_{a}^{c} f(x) dx + \int_{c}^{b} f(x) dx \text{ where } a \le c \le b$$

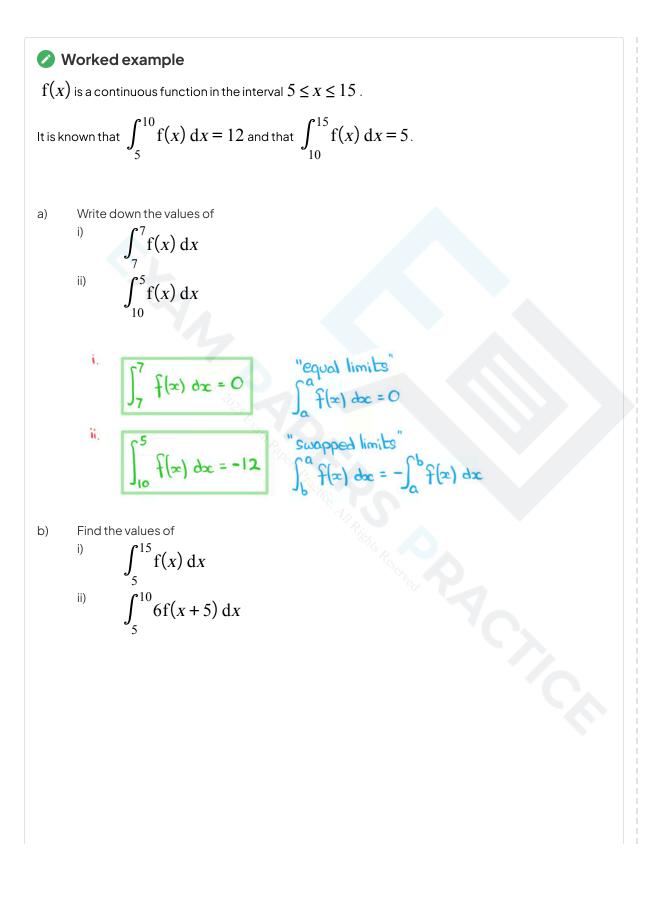
• this is particularly useful for areas under multiple curves or areas under the X-axis

horizontal translations

•
$$\int_{a}^{b} f(x) dx = \int_{a-k}^{b-k} f(x+k) dx$$
 where k is a constant

• the graph of $y = f(x \pm k)$ is a horizontal translation of the graph of y = f(x)(f(x + k) translates left, f(x - k) translates right)







$$i. I = \int_{5}^{16} f(x) dx = \int_{5}^{10} f(x) dx + \int_{10}^{16} f(x) dx = 12 + 5 = 17$$

$$\int_{6}^{16} f(x) dx = \int_{6}^{16} f(x) dx + \int_{6}^{6} f(x) dx$$

$$i: \int_{5}^{15} f(x) dx = 17$$

$$ii. I = \int_{5}^{10} 6f(x+5) = 6 \int_{5}^{10} f(x+5) dx$$

$$\int_{6}^{16} kf(x) dx = k \int_{6}^{10} f(x) dx$$

$$I = 6 \int_{5+5}^{10+5} f(x) dx$$

$$I = 6 \int_{5+5}^{10+5} f(x) dx$$

$$I = 6 \int_{10}^{10+5} f(x) dx = 6 \times 5 = 30$$

$$i: \int_{5}^{10} 6f(x+5) dx = 30$$



5.4.4 Further Applications of Integration

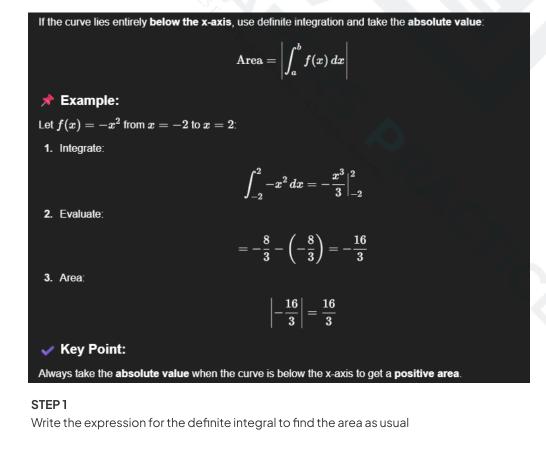
Negative Integrals

- The area under a curve may appear fully or partially under the x-axis
 - This occurs when the function f(x) takes **negative** values within the boundaries of the area
- The definite integrals used to find such areas
 - will be **negative** if the area is **fully** under the *X*-axis
 - possibly **negative** if the area is **partially** under the *X*-axis
 - this occurs if the negative area(s) is/are greater than the positive area(s), their sum will be negative
- When using a GDC use the modulus (absolute value) function so that all definite integrals have a
 positive value

$$A = \int_{a}^{b} |y| \, \mathrm{d}x$$

This is given in the formula booklet

How do I find the area under a curve when the curve is fully under the x-axis?





This may involve finding the lower and upper limits from a graph sketch or GDC and f(x) may need to be rewritten in an integrable form

STEP 2

The answer to the definite integral will be negative Area must always be positive so take the modulus (absolute value) of it e.g. If I = -36 then the area would be 36 (square units)

How do I find the area under a curve when the curve is partially under the x-axis?

• For questions that allow the use of a GDC you can still use

$$A = \int_{a}^{c} |f(x)| \, \mathrm{d}x$$

 To find the area analytically (manually) use the following method STEP 1

Split the area into parts - the area(s) that are above the x-axis and the area(s) that are below the x-axis

STEP 2

Write the expression for the definite integral for each part (give each part a name, l_1, l_2 , etc) This may involve finding the lower and upper limits of each part from a graph sketch or a GDC, finding the roots of the function (i.e. where f(x) = 0) and rewriting f(x) in an integrable form

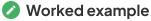
STEP 3

Find the value of each definite integral separately

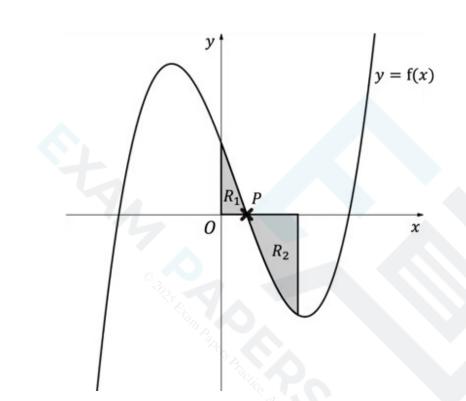
STEP 4

Find the area by summing the modulus (absolute values) of each integral (Mathematically this would be written $A = |I_1| + |I_2| + |I_3| + ...)$





The diagram below shows the graph of y = f(x) where f(x) = (x+4)(x-1)(x-5).



The region R_1 is bounded by the curve y = f(x), the x-axis and the y-axis.

The region R_2 is bounded by the curve y = f(x), the x-axis and the line x = 3.

a) Determine the coordinates of the point labelled P.

```
The x-coordinate of P is a root of f(x) = 0

(x+\psi)(x-1)(x-5)=0

x=-\psi, x=1, x=5

Clearly from the graph, x=1 at point P

\therefore P(1, 0)
```

```
b) i) Find a definite integral that would help find the area of the shaded region R_2 and briefly explain why this would not give the area of the region R_2.
```

ii) Find the exact area of the shaded region R_2 .



```
b) i)

I_{2} = \int_{1}^{3} (x+y)(x-1)(x-5) dx
R<sub>2</sub> is underneath the x-axis so the value of the definite integral will be negative. Area cannot be negative.

i) STEP 1: I_{2} = \int_{1}^{3} (x+y)(x-1)(x-5) dx

I_{3} = \int_{1}^{3} (x^{2}+3x-y)(x-5) dx
Rewrite in an integrable form

I_{2} = \int_{1}^{3} (x^{3}-2x^{2}-19x+20) dx

I_{2} = \left[\frac{x^{4}}{4} - \frac{2x^{3}}{3} - \frac{91x^{2}}{2} + 20x\right]_{1}^{3}
Integrate (no need for "+c")

I_{2} = \left(\frac{3^{4}}{4} - \frac{2(3)^{3}}{3} - \frac{19(2)^{2}}{2} + 20(3)\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{19}{2} + 20\right)

I_{2} = -\frac{93}{4} - \frac{121}{12}

I_{2} = -\frac{100}{3}

STEP 2: Area of R<sub>2</sub>, A<sub>2</sub> = <u>100</u> square units
```

c) Find the exact total area of the shaded regions, R_1 and R_2

```
c) STEP 1, 2: A_1 = I_1 = \int_0^1 \left[ x^3 - 2x^2 - 19x + 20 \right] dx

I_1 = \left[ \frac{x^4}{h} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x \right]_0^1

STEP 3: I_1 = \frac{121}{12} - 0

STEP 4: A_1 + A_2 = \frac{121}{12} + \frac{100}{3} = \frac{521}{12}

\therefore Total area shaded = \frac{521}{12} square units

You can check the final answer using your GOC

and the formula (in booklet) A = \int_a^b |y| dx.

Here, A = \int_a^3 |(x+4)(x-1)(x-5)| dx

A = 43.41666...
```

(Note that our GOC was not able to produce the exact answer...)



Area Between a Curve and a Line

- Areas whose boundaries include a curve and a (non-vertical) straight line can be found using integration
 - For an **area** under a **curve** a **definite integral** will be needed
 - For an **area** under a **line** the shape formed will be a **trapezium** or **triangle**
 - **basic area formulae** can be used rather than a definite integral
 - (although a definite integral would still work)
- The area required could be the sum or difference of areas under the curve and line

How do I find the area between a curve and a line?

STEP 1

If not given, sketch the graphs of the curve and line on the same diagram Use a GDC to help with this step

STEP 2

Find the intersections of the curve and the line If no diagram is given this will help identify the area(s) to be found

STEP 3

Determine whether the area required is the sum or difference of the area under the curve and the area under the line

Calculate the area under a curve using a integral of the form

$$\int_{a}^{b} y \, \mathrm{d}x$$

Calculate the area under a line using either $A = \frac{1}{2}bh$ for a triangle or $A = \frac{1}{2}h(a+b)$ for a

trapezium (y-coordinates will be needed)

STEP 4

Evaluate the definite integrals and find their sum or difference as necessary to obtain the area required

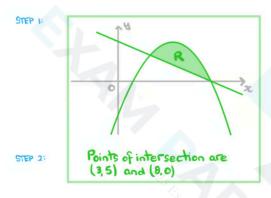


Worked example

The region R is bounded by the curve with equation $y = 10x - x^2 - 16$ and the line with equation y = 8 - x.

R lies entirely in the first quadrant.

a) Using your GDC, or otherwise, sketch the graphs of the curve and the line on the same diagram. Identify and label the region R on your sketch and use your GDC to find the x-coordinates of the points of intersection between the curve and the line.



- b) i) Write down an integral that would find the area of the region R.
 - ii) Find the area of the region R.

1) STEP 3: Curve is import boundary of R

$$\therefore y_{1} = 10x - x^{2} - 16$$

$$y_{2} = 8 - x$$

$$y_{1} - y_{2} = 10x - x^{2} - 16 - (8 - x) = 11x - x^{2} - 24$$

$$\therefore \text{ Area of R, } A_{R} = \int_{3}^{8} (11x - x^{2} - 24) dx$$

$$H_{R} = \int_{3}^{8} (11x - x^{2} - 24) dx$$

$$A_{R} = \begin{bmatrix} 11x^{2} - x^{3} - 24x \end{bmatrix}_{3}^{8}$$

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$$A_{R} = \begin{bmatrix} 11x^{2} - x^{3} - 24x \end{bmatrix}_{3}^{8} - 24x (3) = \begin{bmatrix} 11(2)^{2} - (3)^{3} - 24x (3) \end{bmatrix}$$

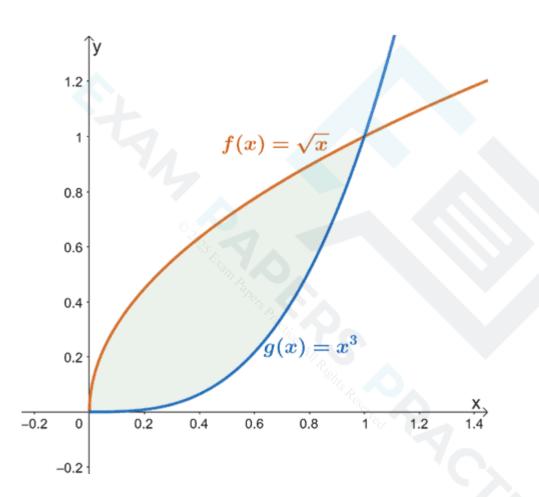
$$A_{R} = -\frac{32}{3} - -\frac{63}{2}$$

$$\therefore \text{ Area of region R is } \underline{125} \text{ square units}$$



Area Between 2 Curves

- Areas whose boundaries include two curves can be found by integration
 - The area between two curves will be the difference of the areas under the two curves
 - both areas will require a **definite integral**
 - Finding points of intersection may involve a more awkward equation than solving for a curve and a line



How do I find the area between two curves?

STEP 1

If not given, sketch the graphs of both curves on the same diagram Use a GDC to help with this step

STEP 2

Find the intersections of the two curves If no diagram is given this will help identify the area(s) to be found



STEP 3

For each area (there may only be one) determine which curve is the 'upper' boundary For each area, write a definite integral of the form

$$\int_{a}^{b} (y_1 - y_2) \,\mathrm{d}x$$

where y_1 is the function for the 'upper' boundary and y_2 is the function for the 'lower' boundary Be careful when there is more than one region – the 'upper' and 'lower' boundaries will swap

STEP 4

Evaluate the definite integrals and sum them up to find the total area (Step 3 means no definite integral will have a negative value)



Worked example

The diagram below shows the curves with equations y = f(x) and y = g(x) where

$$f(x) = (x-2)(x-3)^2$$

$$g(x) = x^2 - 5x + 6$$

Find the area of the shaded region.

