DP IB Maths: AI HL

5.4 Further Integration

Contents

5.4.1 Integrating Special Functions

Integrating Trig Functions

How do I integrate sin, cos and 1/cos²?

- The **antiderivatives** for **sine** and **cosine** are

$$\int \sin\ x\ \mathrm{d}x = -\cos\ x + c$$

$$\int \cos\ x\ \mathrm{d}x = \sin\ x + c$$

where c is the **constant** of **integration**

- Also, from the **derivative** of $\tan x$

$$\int \frac{1}{\cos^2 x}\ \mathrm{d}x = \tan\ x + c$$

- All three of these standard integrals are in the **formula booklet**
- For the **linear** function $ax + b$, where a and b are constants,

$$\int \sin\ (ax+b)\ \mathrm{d}x = -\frac{1}{a}\cos\ (ax+b) + c$$

$$\int \cos\ (ax+b)\ \mathrm{d}x = \frac{1}{a}\sin\ (ax+b) + c$$

$$\int \frac{1}{\cos^2 (ax+b)}\ \mathrm{d}x = \frac{1}{a}\tan(ax+b) + c$$

- For **calculus** with **trigonometric** functions **angles must** be **measured** in **radians**
  - Ensure you know how to change the angle mode on your GDC

Worked example

a) Find, in the form $\mathrm{F}(x)+c$, an expression for each integral

i. $\int \cos x \, \mathrm{d}x$

ii. $\int \frac{1}{\cos^2\left(3x-\frac{\pi}{3}\right)} \, \mathrm{d}x$

i. $\int \cos x \, dx = \sin x + c$

ii. $\int \frac{1}{\cos^2\left(3x-\frac{\pi}{3}\right)} \, dx = \frac{1}{3}\tan\left(3x-\frac{\pi}{3}\right)+c$

(Linear function $ax+b$)

b) A curve has equation $y=\int 2\sin\left(2x+\frac{\pi}{6}\right) \mathrm{d}x$. The curve passes through the point with coordinates $\left(\frac{\pi}{3}, \sqrt{3}\right)$.

Find an expression for y.

$$y = 2\int \sin\left(2x + \frac{\pi}{6}\right) dx$$

$$y = 2\left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{6}\right)\right] + c$$

At $x = \frac{\pi}{3}$, $y = \sqrt{3}$, $\sqrt{3} = -\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) + c$

$$c = \cos\left(\frac{5\pi}{6}\right) + \sqrt{3}$$

$$c = \frac{\sqrt{3}}{2}$$

$$\therefore y = \frac{\sqrt{3}}{2} - \cos\left(2x + \frac{\pi}{6}\right)$$

Integrating e^x & 1/x

How do I integrate exponentials and 1/x?

- The **antiderivatives** involving e^x and $\ln x$ are

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

where c is the **constant** of **integration**

  - These are given in the **formula booklet**
- For the **linear** function $(ax + b)$, where a and b are constants,

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + c$$

- It follows from the last result that

$$\int \frac{a}{ax+b} \, dx = \ln|ax+b| + c$$

  - which can be deduced using **Reverse Chain Rule**
- With **ln,** it can be useful to write the constant of integration, c, as a logarithm
  - using the laws of logarithms, the answer can be written as a single term
  - $\int \frac{1}{x} \, dx = \ln|x| + \ln k = \ln k|x|$ where k is a constant
  - This is similar to the special case of **differentiating** $\ln(ax + b)$ when $b = 0$

Worked example

A curve has the gradient function $f'(x) = \dfrac{3}{3x+2} + e^{4-x}$.

Given the exact value of $f(1)$ is $\ln 10 - e^3$ find an expression for $f(x)$.

$$f(x) = \int \left(\frac{3}{3x+2} + e^{4-x} \right) dx$$

$$f(x) = 3\int \frac{1}{3x+2}\, dx + \int e^{4-x}\, dx$$

$$= 3\left[\frac{1}{3} \ln |3x+2| \right] - e^{4-x} + c$$

$$f(1) = \ln 10 - e^3, \quad \ln|3 \times 1 + 2| - e^{4-1} + c = \ln 10 - e^3$$

$$\therefore c = \ln 10 - \ln 5$$

$$c = \ln\left(\frac{10}{5}\right) = \ln 2$$

$$\therefore f(x) = \ln|3x+2| - e^{4-x} + \ln 2$$

$$\left(= \ln 2|3x+2| - e^{4-x}\right)$$

5.4.2 Techniques of Integration

Integrating Composite Functions (ax+b)

What is a composite function?

- A **composite function** involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being **linear** – i.e. of the form $ax + b$

How do I integrate linear (*ax+b*) functions?

- A **linear function** (of x) is of the form $ax + b$
- The special cases for **trigonometric functions** and **exponential** and **logarithm functions** are
  - $\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$
  - $\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$
  - $\int \mathrm{e}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b} + c$
  - $\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$
- There is one more special case
  - $\int (ax+b)^n\,\mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$ where $n \in \mathbb{Q},\ n \neq -1$
- c, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
  - However, spotting them can make solutions more efficient

Worked example

Find the following integrals

a) $\int 3(7-2x)^{\frac{5}{3}}\,\mathrm{d}x$

$I = \int 3(7-2x)^{5/3}\,dx = 3\int (-2x+7)^{5/3}\,dx$

Using $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$,

$I = 3\left[\frac{1}{-2 \times \frac{8}{3}}(-2x+7)^{8/3}\right] + c$ ← $5/3 + 1$

$\therefore I = -\frac{9}{16}(7-2x)^{8/3} + c$

b) $\int \frac{1}{2}\cos(3x-2)\,\mathrm{d}x$

$I = \int \frac{1}{2}\cos(3x-2)\,dx = \frac{1}{2}\int \cos(3x-2)\,dx$

Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

$I = \frac{1}{2}\left[\frac{1}{3}\sin(3x-2)\right] + c$

$\therefore I = \frac{1}{6}\sin(3x-2) + c$

Reverse Chain Rule

What is reverse chain rule?

- The **Chain Rule** is a way of differentiating two (or more) functions
- **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
  - spotting that chain rule would be used in the reverse (differentiating) process

How do I know when to use reverse chain rule?

- **Reverse chain rule** is used when we have the **product** of a **composite function** and the **derivative** of its **secondary function**
- Integration is trickier than differentiation; many of the shortcuts do not work
  - For example, in general $\int e^{f(x)}\,\mathrm{d}x \neq \frac{1}{f'(x)}e^{f(x)}$
  - However, this result **is true** if $f(x)$ is linear $(ax+b)$
- Formally, in **function notation**, **reverse chain rule** is used for **integrands** of the form

$$I = \int g'(x) f'(g(x))\,\mathrm{d}x$$

  - this does not have to be strictly true, but '*algebraically*' it should be
    - if **coefficients** do not match '**adjust** and **compensate'** can be used
    - e.g. $5x^2$ is not quite the derivative of $4x^3$
      - the *algebraic* part (x^2) is *'correct'*
      - but the coefficient 5 is '*wrong*'
      - use '**adjust** and **compensate'** to '*correct'* it
- A particularly useful instance of reverse chain rule to recognise is

$$I = \int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$$

  - i.e. the **numerator** is (almost) the **derivative** of the **denominator**
  - '**adjust** and **compensate**' may need to be used to deal with any coefficients
    - e.g.

$$I = \int \frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\int 3\frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\ln|x^3+3x| + c$$

How do I integrate using reverse chain rule?

- If the product **can** be identified, the **integration** can be done "by **inspection**"
  - there may be some "**adjusting** and **compensating**" to do
- Notice a lot of the "**adjust** and **compensate** method" happens mentally
  - this is indicated in the steps below by quote marks

STEP 1
Spot the 'main' function

e.g. $I = \int x(5x^2 - 2)^6 \, dx$

"the main function is $(\ ...\ )^6$ which would come from $(\ ...\ )^7$"

STEP 2

'**Adjust'** and '**compensate'** any coefficients required in the integral

e.g. " $(\ ...\ )^7$ would differentiate to $7(\ ...\ )^6$"

"chain rule says multiply by the derivative of $5x^2 - 2$, which is $10x$"

"there is no '7' or '10' in the integrand so adjust and compensate"

$$I = \frac{1}{7} \times \frac{1}{10} \times \int 7 \times 10 \times x(5x^2 - 2)^6 \, dx$$

STEP 3

Integrate and simplify

e.g. $I = \frac{1}{7} \times \frac{1}{10} \times (5x^2 - 2)^7 + c$

$$I = \frac{1}{70}(5x^2 - 2)^7 + c$$

- Differentiation can be used as a means of checking the final answer
- After some practice, you may find Step 2 is not needed
  - Do use it on more awkward questions (negatives and fractions!)
- If the product **cannot** easily be identified, use **substitution**

Worked example

A curve has the gradient function $f'(x) = 5x^2\sin(2x^3)$.

Find an expression for $f(x)$.

$$f(x) = \int 5x^2 \sin(2x^3)\,dx$$

$$f(x) = 5\int x^2 \sin(2x^3)\,dx$$ Take 5 out as a factor

This is a product, <u>almost</u> in the form $g'(x)f(g(x))$

STEP 1: Spot the 'main' function

"the main function is sin (...) which would come from cos (...)"

STEP 2: 'Adjust and compensate' coefficients

"cos (...) would differentiate to −sin (...)"

"$2x^3$ would differentiate to $6x^2$"

$$f(x) = 5 \times - \times \frac{1}{6} \times \int - \times 6 \times x^2 \sin(2x^3)\,dx$$

↑ ↑ compensate ↑ ↑ adjust

STEP 3: Integrate and simplify

$$f(x) = -\frac{5}{6}\cos(2x^3) + c$$

Substitution: Reverse Chain Rule

What is integration by substitution?

- When reverse chain rule is difficult to spot or awkward to use then **integration** by **substitution** can be used
  - **substitution** simplifies the integral by defining an alternative variable (usually u) in terms of the original variable (usually x)
  - **everything** (including "$\mathrm{d}x$" and **limits** for **definite integrals**) is then substituted which makes the integration much easier

How do I integrate using substitution?

STEP 1
Identify the substitution to be used – it will be the secondary function in the composite function

So $g(x)$ in $f(g(x))$ and $u = g(x)$

STEP 2
Differentiate the substitution and rearrange

$\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction

(i.e. "multiply by $\mathrm{d}x$" to get rid of fractions)

STEP 3
Replace all parts of the integral
All x terms should be replaced with equivalent u terms, including $\mathrm{d}x$
If finding a **definite integral** change the limits from x-values to u-values too

STEP 4
Integrate and either

substitute x back in

or

evaluate the definte integral using the u limits (either using a GDC or manually)

STEP 5
Find c, the constant of integration, if needed

- For **definite integrals**, a GDC should be able to process the integral without the need for a substitution
  - be clear about whether working is required or not in a question

Worked example

a) Find the integral

$$\int \frac{6x+5}{(3x^2+5x-1)^3}\,\mathrm{d}x$$

STEP 1: Identify the substitution

The composite function is $(3x^2+5x-1)^3$

The secondary function of this is $3x^2+5x-1$

$\therefore$ Let $u = 3x^2+5x-1$

STEP 2: Differentiate u and rearrange

$$\frac{du}{dx} = 6x+5$$

$$\therefore\ du = (6x+5)\,dx$$

STEP 3: Replace all parts of the integral

$$I = \int \frac{6x+5}{(3x^2+5x-1)^2}\,dx = \int \frac{du}{u^3}$$

$$= \int u^{-3}\,du$$

STEP 4: Integrate and substitute x back in

(STEP 5 not needed, evaluating c is not required)

$$I = \frac{u^{-2}}{-2} + c$$

$$I = -\frac{1}{2}(3x^2+5x-1)^{-2} + c$$

$$\therefore\ I = \frac{-1}{2(3x^2+5x-1)^2} + c$$

b) Evaluate the integral

$$\int_1^2 \frac{6x+5}{(3x^2+5x-1)^3}\,dx$$

giving your answer as an exact fraction in its simplest terms.

Note that you could use your GDC for this part
Certainly use it to check your answer!

From STEP 3 above, $I = \int_{x=1}^{x=2} u^{-3}\,du$

Change limits too, $x=1,\ u=3(1)^2+5(1)-1=7$

$x=2,\ u=3(2)^2+5(2)-1=21$

STEP 4: Integrate and evaluate

$I = \left[-\frac{1}{2}u^{-2}\right]_7^{21} = \left[-\frac{1}{2}(21)^{-2}\right] - \left[-\frac{1}{2}(7)^{-2}\right]$

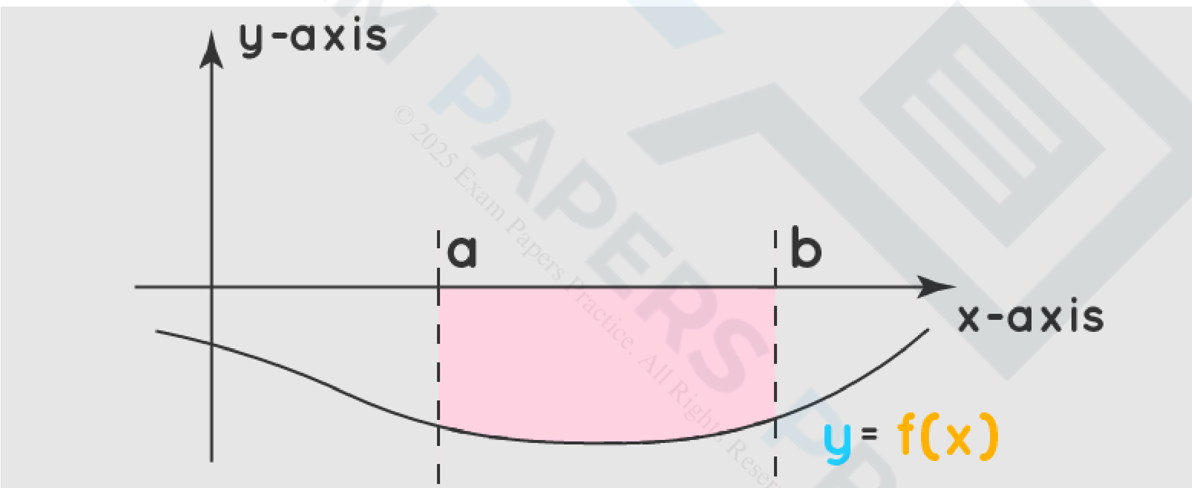
$\therefore I = \frac{4}{441}$

5.4.3 Further Applications of Integration

Negative Integrals

- The area under a curve may appear **fully** or **partially** under the x-axis
  - This occurs when the function $f(x)$ takes **negative** values within the boundaries of the area
- The **definite integrals** used to find such **areas**
  - will be **negative** if the area is **fully** under the x-axis
  - possibly **negative** if the area is **partially** under the x-axis
    - this occurs if the negative area(s) is/are greater than the positive area(s), their **sum** will be **negative**

How do I find the area under a curve when the curve is fully under the x-axis?



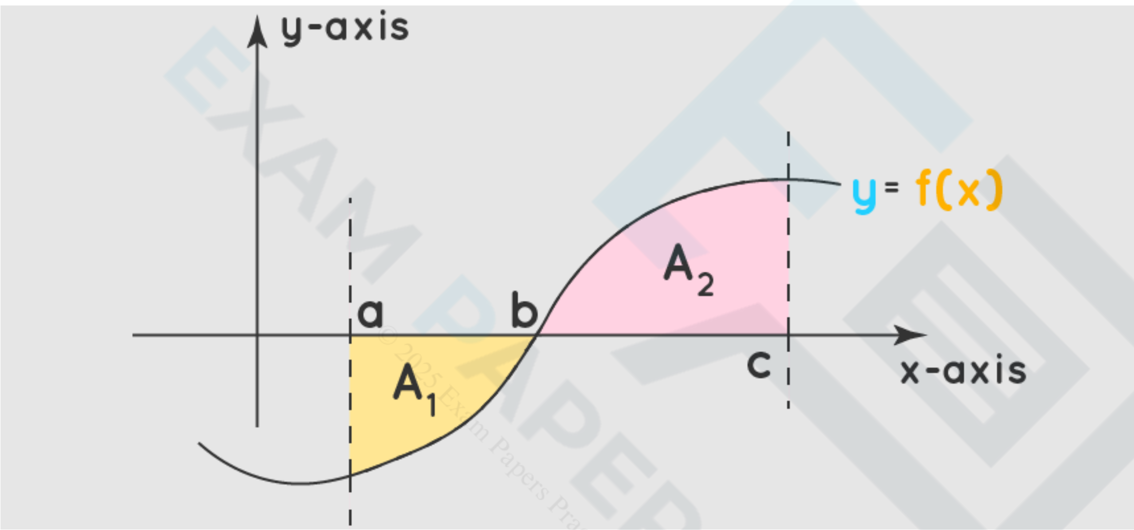
STEP 1
Write the expression for the definite integral to find the area as usual
This may involve finding the lower and upper limits from a graph sketch or GDC and f(x) may need to be rewritten in an integrable form

STEP 2
The answer to the definite integral will be negative
Area must always be positive so take the modulus (absolute value) of it

e.g. If $I = -36$ then the area would be 36 (square units)

How do I find the area under a curve when all, or some, of the curve is below the x-axis?

- Use the **modulus** function
  - The **modulus** is also called the **absolute value** (Abs)
  - Essentially the modulus function makes **all** function **values positive**
  - Graphically, this means any negative areas are reflected in the x-axis



- A GDC will recognise the modulus function
  - look for a key or on-screen icon that says 'Abs' (absolute value)

$$A = \int_a^b |y|\, dx$$

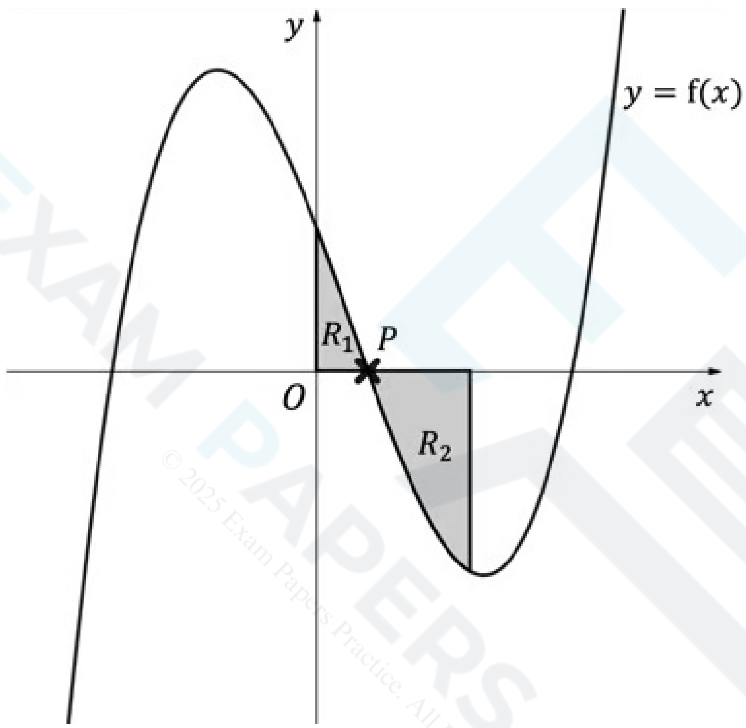
  - This is given in the **formula booklet**

STEP 1
If a diagram is not given, use a GDC to draw the graph of $y = f(x)$
If not identifiable from the question, use the graph to find the limits a and b

STEP 2
Write down the definite integral needed to find the required area
Remember to include the modulus (| ... |) symbols around the function
Use the GDC to evaluate it

Worked example

The diagram below shows the graph of $y = f(x)$ where $f(x) = (x+4)(x-1)(x-5)$.



The region R_1 is bounded by the curve $y = f(x)$, the x-axis and the y-axis.

The region R_2 is bounded by the curve $y = f(x)$, the x-axis and the line $x = 3$.

Find the total area of the shaded regions, R_1 and R_2.

STEP 1: Graph given, identify limits

$a = 0$ (y-axis)

$b = 3$ (line $x = 3$)

STEP 2: Write down the integral required and use a GDC to evaluate it

$$A = \int_0^3 \left| (x+4)(x-1)(x-5) \right| \, dx$$

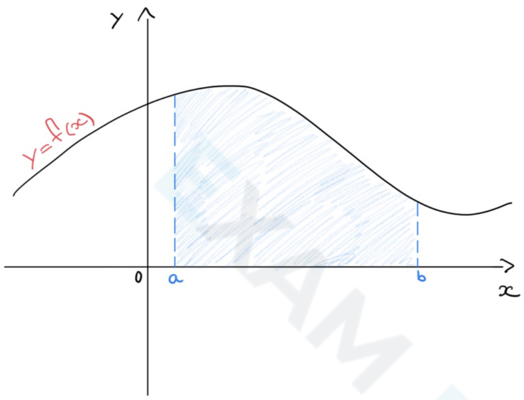
$A = 43.166\ 666 \ldots$

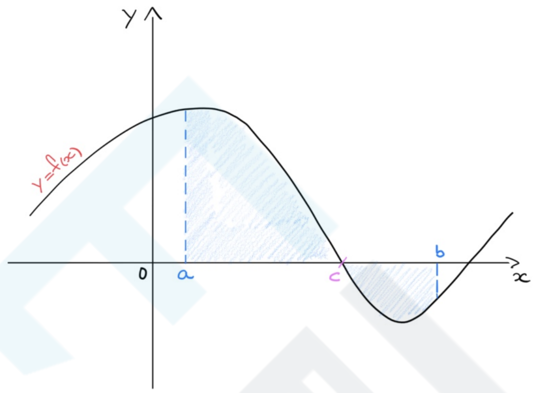
$\therefore A = 43.2$ square units (3 s.f.)

Area Between Curve & y-axis

Your notes

What is meant by the area between a curve and the y-axis?





- The **area** referred to is the **region bounded** by
  - the graph of $y = f(x)$
  - the y-axis
  - the **horizontal** line $y = a$
  - the **horizontal** line $y = b$
- The **exact area** can be found by evaluating a **definite integral**

How do I find the area between a curve and the y-axis?

- Use the formula

$$A = \int_a^b |x| \, dy$$

  - This is given in the **formula booklet**
  - The function is normally given in the form $y = f(x)$
    - so will need rearranging into the form $x = g(y)$
  - a and b may not be given directly as could involve the the x-axis ($y = 0$) and/or a root of $x = g(y)$
    - use a GDC to plot the curve and find roots as necessary

STEP 1

If a diagram is not given, use a GDC to draw the graph of $y = f(x)$ (or $x = g(y)$ if already in that form)

If not identifiable from the question, use the graph to find the limits a and b

STEP 2

If needed, rearrange $y = f(x)$ into the form $x = g(y)$

STEP 3

Write down the definite integral needed to find the required area

Use a GDC to evaluate it

A GDC is likely to require the function written with 'x' as the variable (not 'y')

Remember to include the modulus (| ... |) symbols around the function

Modulus may be called 'Absolute value (Abs)' on some GDCs

- In trickier problems some (or all) of the area may be 'negative'
  - this would be any area that is to the **left** of the y-axis (negative x values)
  - $|x|$ makes such areas 'positive' by **reflecting** them in the y-axis
    - a GDC will apply $|x|$ automatically as long as the modulus (| ... |)symbols are included

Worked example

Find the area enclosed by the curve with equation $y = 2 + \sqrt{x + 4}$, the y-axis and the horizontal lines with equations $y = 3$ and $y = 6$.

STEP 1: GDC plot shows partially negative area; limits given in question

From GDC.

STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = 2 + \sqrt{x+4}$

$x = (y-2)^2 - 4 = y^2 - 4y + 4 - 4$

$x = y^2 - 4y$

STEP 3: Write down integral; use GDC to evaluate

$A = \int_3^6 |y^2 - 4y| \, dy$

(Type this as $\int_3^6 |x^2 - 4x| \, dx$ on a GDC)

might be 'Abs' on a GDC

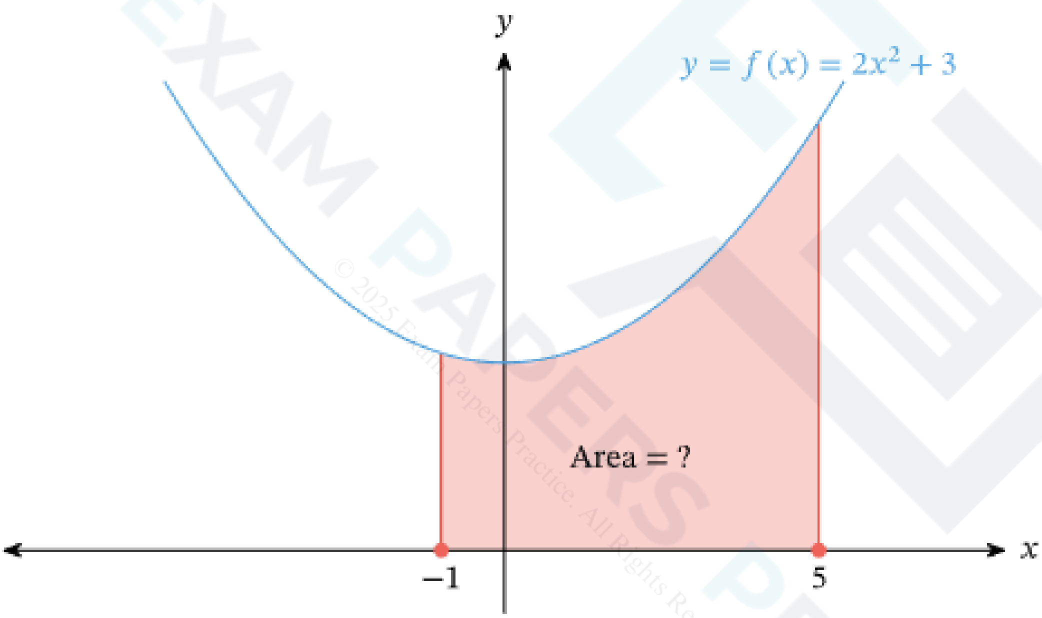
$A = 12.333\ 333...$

$\therefore A = 12.3$ square units (3 s.f.)

The exact answer is $^{37}/_3$ but our GDC was not able to recognise this, despite us trying to use the exact-approximate button (S-D). This may vary between makes/models and will be due to the algorithm used to calculate integrals.

Area Between a Curve and a Line

- **Areas** whose boundaries include a **curve** and a (non-vertical) **straight line** can be found using integration
  - For an **area** under a **curve** a **definite integral** will be needed
  - For an **area** under a **line** the shape formed will be a **trapezium** or **triangle**
    - **basic area formulae** can be used rather than a definite integral
    - using a GDC, one method is not particularly trickier than the other
- The **total area** required could be the **sum** or **difference** of the area under the curve and the area under the line



How do I find the area between a curve and a line?

STEP 1
If a diagram is not given, use a GDC to draw the graphs of the curve and line and identify the area to be found

STEP 2
Use a GDC to find the root(s) of the curve, the root of the line, and the *x*-coordinates of any intersections between the curve and the line.

STEP 3
Use the graph to determine whether areas will need adding or subtracting
Deduce the limits and thus the definite integral(s) to find the area(s) under the curve and the line
Use a GDC to calculate the area under the curve

$$\int_a^b |y| \, dx$$

Remember to include the modulus (|...|) symbols around the function
Use a GDC to calculate the area under the line - this could be another definite integral or

$A = \frac{1}{2}bh$ for a triangle or $A = \frac{1}{2}h(a + b)$ for a trapezium

STEP 4
Add or subtract areas accordingly to obtain a final answer

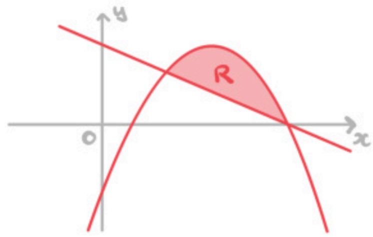
Worked example

The region R is bounded by the curve with equation $y = 10x - x^2 - 16$ and the line with equation $y = 8 - x$.

R lies entirely in the first quadrant.

Find the area of the region *R*.

STEP 1: Sketch the graph from GDC plot; identify area required



STEP 2: Only intersections are required (use GDC)

Points of intersection are
(3, 5) and (8, 0)

STEP 3: Determine +/−, limits, integrals, etc

Area under curve $= \int_3^8 |10x - x^2 - 16|\, dx = \frac{100}{3}$

Area under line $= \frac{1}{2} \times (8-3) \times 5 = \frac{25}{2}$

$\therefore$ Area of R $= \frac{100}{3} - \frac{25}{2} = \frac{125}{6}$

Area of R $= \frac{125}{6}$ square units (20.8 3 s.f.)

If finding the area of R directly from your GDC you may find it will not give an exact answer. In this case, an exact answer was not demanded so either 125/6 or 20.8 (3 s.f.) is acceptable

Definite Integrals

What is a definite integral?

$$\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- ***a*** and ***b*** are called limits
  - ***a*** is the lower limit
  - ***b*** is the upper limit
- $f(x)$ is the **integrand**
- $F(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** ("**+c**") is not needed in **definite integration**
  - "+c" would appear alongside both **F(*a*)** and **F(*b*)**
  - subtracting means the "+c"'s cancel

How do I find definite integrals analytically (manually)?

STEP 1
Give the integral a name to save having to rewrite the whole integral every time
If need be, rewrite the integral into an integrable form

$$I = \int_a^b f(x)\,dx$$

STEP 2
Integrate without applying the limits; you will not need "+c"
Notation: use square brackets [ ] with limits placed at the end bracket

STEP 3
Substitute the limits into the function and evaluate

Worked example

a) Show that

$$\int_2^4 3x(x^2-2)\ \mathrm{d}x = 144$$

STEP 1: Name the integral and rewrite into an integratable form

$$I = \int_2^4 (3x^3 - 6x)\ dx$$

STEP 2: Integrate

$$I = \left[\frac{3}{4}x^4 - 3x^2\right]_2^4$$

STEP 3: Evaluate

$$I = \left[\frac{3}{4}(4)^4 - 3(4)^2\right] - \left[\frac{3}{4}(2)^4 - 3(2)^2\right]$$

$$I = 144 - 0$$

$$\therefore \int_2^4 3x(x^2-2) = 144$$

b) Use your GDC to evaluate

$$\int_0^1 3e^{x^2 \sin x}\ \mathrm{d}x$$

giving your answer to three significant figures.

Using GDC,

$$\int_0^1 3e^{x^2 \sin x}\, dx = 3.872\,957\ldots$$

$$\therefore \int_0^1 3e^{x^2 \sin x}\, dx = 3.87 \quad (3 \text{ s.f.})$$

5.4.4 Volumes of Revolution

Volumes of Revolution Around x-axis

What is a volume of revolution around the x-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the x-axis
- The **volume** of **revolution** is the volume of this solid
- Be careful – the 'front' and 'back' of this solid are flat
  - they were created from straight (vertical) lines
  - 3D sketches can be misleading

How do I solve problems involving the volume of revolution around the x-axis?

- Use the formula

$$V = \pi \int_a^b y^2 \, dx$$

  - This is given in the **formula booklet**
  - y is a function of x
  - $x = a$ and $x = b$ are the equations of the (vertical) lines bounding the area
    - If $x = a$ and $x = b$ are not stated in a question, the boundaries could involve the y-axis ($x = 0$) and/or a root of $y = f(x)$
    - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1
If a diagram is not given, use a GDC to draw the graph of $y = f(x)$
If not identifiable from the question, use the graph to find the limits a and b

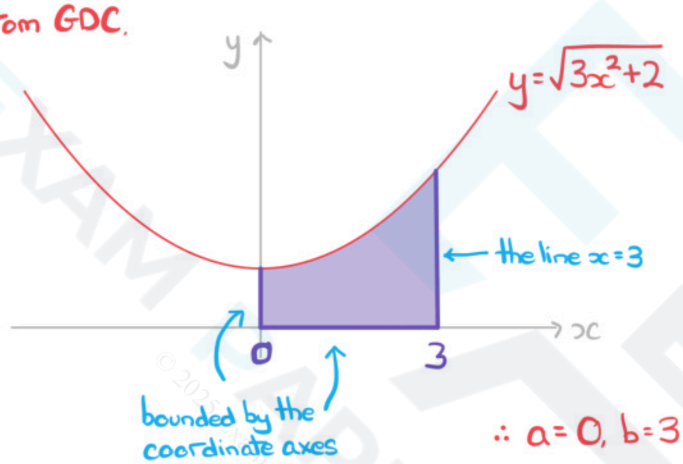
STEP 2
Use a GDC and the formula to evaluate the integral
Thus find the volume of revolution

Worked example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \sqrt{3x^2 + 2}$, the coordinate axes and the line $x = 3$ by 2π radians around the x-axis. Give your answer as an exact multiple of π.

STEP 1: Use GDC to plot $y = f(x)$; identify limits

From GDC.



$\therefore a = 0, b = 3$

STEP 2: Use GDC and formula, find volume

$$V = \pi \int_0^3 \left(\sqrt{3x^2+2}\right)^2 \, dx = 33\pi$$

$\therefore V = 33\pi$ cubic units (104 3s.f.)

Depending on make/model of your GDC you may or may not get an exact answer. If you don't, try evaluating the integral without π (but remember to put it back for your written answer!)

Volumes of Revolution Around y-axis

What is a volume of revolution around the y-axis?

- Very similar to above, this is a **solid** of **revolution** which is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the y-axis
- The **volume** of **revolution** is the volume of this solid

How do I solve problems involving the volume of revolution around y-axis?

- Use the formula

$$V = \pi \int_a^b x^2 \, dy$$

  - This is given in the **formula booklet**
  - x is a function of y
    - the function is usually given in the form $y = f(x)$
    - this will need rearranging into the form $x = g(y)$
  - $y = a$ and $y = b$ are the equations of the (horizontal) lines bounding the area
    - If $y = a$ and $y = b$ are not stated in the question, the boundaries could involve the x-axis ($y = 0$) and/or a root of $x = g(y)$
    - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1

If a diagram is not given, use a GDC to draw the graph of $y = f(x)$
(or $x = g(y)$ if already in that form)
If not identifiable from the question use the graph to find the limits a and b

STEP 2

If needed, rearrange $y = f(x)$ into the form $x = g(y)$

STEP 3

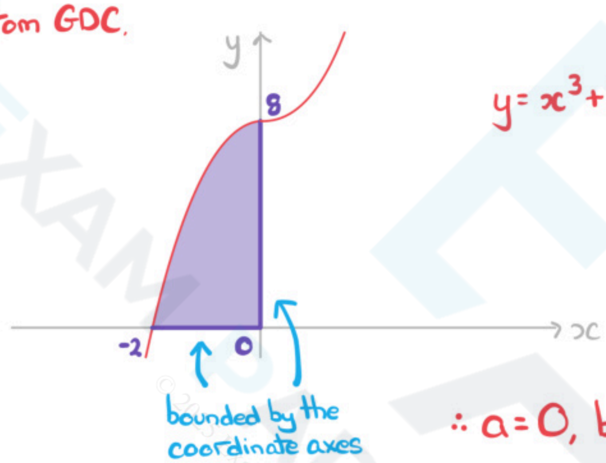
Use a GDC and the formula to evaluate the integral
A GDC will likely require the function written with 'x' as the variable (not 'y')
Thus find the volume of revolution

Worked example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = x^3 + 8$ and the coordinate axes by 2π radians around the y-axis. Give your answer to three significant figures.

STEP 1: Use GDC to plot $y=f(x)$; identify limits

From GDC.



$\therefore a=0,\ b=8$

STEP 2: Rearrange $y=f(x)$ into $x=g(y)$

$y = x^3+8$

$x^3 = y-8$

$x = \sqrt[3]{y-8}$

STEP 3: Use GDC and formula, find volume

$V = \pi \int_0^8 \left(\sqrt[3]{y-8}\right)^2 \, dy$ (Type as $\left(\sqrt[3]{x-8}\right)^2$ on GDC)

$V = 60.318\,578\ldots$

$\therefore V = 60.3$ cubic units (3 s.f.)