DP IB Maths: AA SL

5.4 Further Integration

Contents

5.4.1 Integrating Special Functions

Integrating Trig Functions

How do I integrate sin and cos?

- The **antiderivatives** for **sine** and **cosine** are

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

where c is the **constant** of **integration**

  - These are given in the **formula booklet**
- For the **linear** function $ax + b$, where a and b are constants,

$$\int \sin (ax + b) \, dx = -\frac{1}{a} \cos (ax + b) + c$$

$$\int \cos (ax + b) \, dx = \frac{1}{a} \sin (ax + b) + c$$

- For **calculus** with **trigonometric** functions **angles must** be **measured** in **radians**
  - Ensure you know how to change the angle mode on your GDC

Worked example

a) Find, in the form $\mathrm{F}(x) + c$, an expression for each integral

i) $$\int \cos x \, dx$$

ii) $$\int 3\sin\left(2x + \frac{\pi}{3}\right) dx$$

iii) $$\int (2\sin(4x) - 3\cos(2x)) \, dx$$

i. $$\boxed{\int \cos x \, dx = \sin x + c}$$

ii. $$I = \int 3\sin\left(2x + \frac{\pi}{3}\right) dx = 3\int \sin\left(2x + \frac{\pi}{3}\right) dx$$
$$= 3\left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)\right] + c$$

$$\boxed{\therefore \ I = -\frac{3}{2}\cos\left(2x + \frac{\pi}{3}\right) + c}$$

iii. $$I = \int (2\sin(4x) - 3\cos(2x)) \, dx = 2\int \sin(4x) \, dx - 3\int \cos(2x) \, dx$$
$$= 2\left[-\frac{1}{4}\cos(4x)\right] - 3\left[\frac{1}{2}\sin(2x)\right] + c$$

$$\boxed{\therefore \ I = -\frac{1}{2}\cos(4x) - \frac{3}{2}\sin(2x) + c}$$

Check by differentiating: $-\frac{1}{2}\left[-4\sin(4x)\right] - \frac{3}{2}\left[2\cos(2x)\right]$
$= 2\sin(4x) - 3\cos(2x)$ ✓

b) The graph of $y = \mathrm{F}(x) + c$ for question (a) part (ii) passes through the point with coordinates $\left(\frac{\pi}{3}, \frac{5}{2}\right)$.

Find the value of c.

Part (a) (ii) $\boxed{I = -\frac{3}{2}\cos\left(2x + \frac{\pi}{3}\right) + c}$

$$\therefore \frac{5}{2} = -\frac{3}{2}\cos\left(\underbrace{\cancel{2 \times \frac{\pi}{3} + \frac{\pi}{3}}}_{\pi}\right) + c$$

$$\frac{5}{2} = \frac{3}{2} + c \qquad (\cos\pi = -1)$$

$$\boxed{\therefore c = 1}$$

Integrating e^x & 1/x

How do I integrate exponentials and 1/x?

- The **antiderivatives** involving e^x and $\ln x$ are

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

where c is the **constant** of **integration**

  - These are given in the **formula booklet**
- For the **linear** function $(ax+b)$, where a and b are constants,

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + c$$

- It follows from the last result that

$$\int \frac{a}{ax+b} \, dx = \ln|ax+b| + c$$

  - which can be deduced using **Reverse Chain Rule**
- With **ln,** it can be useful to write the constant of integration, c, as a logarithm
  - using the laws of logarithms, the answer can be written as a single term
  - $\int \frac{1}{x} \, dx = \ln|x| + \ln k = \ln k|x|$ where k is a constant
  - This is similar to the special case of **differentiating** $\ln(ax+b)$ when $b = 0$

Worked example

A curve has the gradient function $f'(x) = \dfrac{3}{3x+2} + e^{4-x}$.

Given the exact value of $f(1)$ is $\ln 10 - e^3$ find an expression for $f(x)$.

$$f(x) = \int \left(\frac{3}{3x+2} + e^{4-x} \right) dx$$

$$f(x) = 3\int \frac{1}{3x+2}\, dx + \int e^{4-x}\, dx$$

$$= 3\left[\frac{1}{3} \ln |3x+2| \right] - e^{4-x} + c$$

$$f(1) = \ln 10 - e^3, \quad \ln |3 \times 1 + 2| - \cancel{e^{4-1}} + c = \ln 10 - \cancel{e^3}$$

$$\therefore c = \ln 10 - \ln 5$$

$$c = \ln\left(\frac{10}{5}\right) = \ln 2$$

$$\boxed{\begin{array}{l} \therefore f(x) = \ln |3x+2| - e^{4-x} + \ln 2 \\ \left(= \ln 2|3x+2| - e^{4-x} \right) \end{array}}$$

5.4.2 Techniques of Integration

Integrating Composite Functions (ax+b)

What is a composite function?

- A **composite function** involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being **linear** – i.e. of the form $\boldsymbol{ax + b}$

How do I integrate linear (*ax+b*) functions?

- A **linear function** (of x) is of the form $ax + b$
- The special cases for **trigonometric functions** and **exponential** and **logarithm functions** are
  - $\int \sin(ax + b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax + b) + c$
  - $\int \cos(ax + b)\,\mathrm{d}x = \frac{1}{a}\sin(ax + b) + c$
  - $\int \mathrm{e}^{ax + b}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax + b} + c$
  - $\int \frac{1}{ax + b}\,\mathrm{d}x = \frac{1}{a}\ln|ax + b| + c$
- There is one more special case
  - $\int (ax + b)^n\,\mathrm{d}x = \frac{1}{a(n + 1)}(ax + b)^{n + 1} + c$ where $n \in \mathbb{Q},\ n \neq -1$
- c, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
  - However, spotting them can make solutions more efficient

Worked example

Find the following integrals

a) $\int 3(7-2x)^{\frac{5}{3}}\,\mathrm{d}x$

$I = \int 3(7-2x)^{5/3}\,dx = 3\int(-2x+7)^{5/3}\,dx$

Using $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,$

$I = 3\left[\frac{1}{-2\times\frac{8}{3}}(-2x+7)^{8/3}\right] + c$ $\quad 5/3+1$

$\boxed{\therefore I = -\frac{9}{16}(7-2x)^{8/3} + c}$

b) $\int \frac{1}{2}\cos(3x-2)\,\mathrm{d}x$

$I = \int \frac{1}{2}\cos(3x-2)\,dx = \frac{1}{2}\int\cos(3x-2)\,dx$

Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

$I = \frac{1}{2}\left[\frac{1}{3}\sin(3x-2)\right] + c$

$\boxed{\therefore I = \frac{1}{6}\sin(3x-2) + c}$

Reverse Chain Rule

What is reverse chain rule?

- The **Chain Rule** is a way of differentiating two (or more) functions
- **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
  - spotting that chain rule would be used in the reverse (differentiating) process

How do I know when to use reverse chain rule?

- **Reverse chain rule** is used when we have the **product** of a **composite function** and the **derivative** of its **secondary function**
- Integration is trickier than differentiation; many of the shortcuts do not work
  - For example, in general $\int e^{f(x)}\, dx \neq \frac{1}{f'(x)} e^{f(x)}$
  - However, this result **is true** if $f(x)$ is linear $(ax + b)$
- Formally, in **function notation**, **reverse chain rule** is used for **integrands** of the form

$$I = \int g'(x) f'(g(x))\, dx$$

  - this does not have to be strictly true, but '*algebraically*' it should be
    - if **coefficients** do not match '**adjust** and **compensate'** can be used
    - e.g. $5x^2$ is not quite the derivative of $4x^3$
      - the *algebraic* part (x^2) is '*correct*'
      - but the coefficient 5 is '*wrong*'
      - use '**adjust** and **compensate'** to '*correct*' it
- A particularly useful instance of reverse chain rule to recognise is

$$I = \int \frac{f'(x)}{f(x)}\, dx = \ln |f(x)| + c$$

  - i.e. the **numerator** is (almost) the **derivative** of the **denominator**
  - '**adjust** and **compensate**' may need to be used to deal with any coefficients
    - e.g.

$$I = \int \frac{x^2+1}{x^3+3x}\, dx = \frac{1}{3}\int 3\frac{x^2+1}{x^3+3x}\, dx = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x}\, dx = \frac{1}{3}\ln |x^3+3x| + c$$

How do I integrate using reverse chain rule?

- If the product **can** be identified, the **integration** can be done "by **inspection**"
  - there may be some "**adjusting** and **compensating**" to do
- Notice a lot of the "**adjust** and **compensate** method" happens mentally
  - this is indicated in the steps below by quote marks

STEP 1
Spot the 'main' function

e.g. $I = \int x(5x^2 - 2)^6 \, dx$

"the main function is $(\ldots)^6$ which would come from $(\ldots)^7$"

STEP 2

'**Adjust'** and '**compensate'** any coefficients required in the integral

e.g. " $(\ldots)^7$ would differentiate to $7(\ldots)^6$"

"chain rule says multiply by the derivative of $5x^2 - 2$, which is $10x$"

"there is no '7' or '10' in the integrand so adjust and compensate"

$$I = \frac{1}{7} \times \frac{1}{10} \times \int 7 \times 10 \times x(5x^2 - 2)^6 \, dx$$

STEP 3

Integrate and simplify

e.g. $I = \frac{1}{7} \times \frac{1}{10} \times (5x^2 - 2)^7 + c$

$$I = \frac{1}{70}(5x^2 - 2)^7 + c$$

- Differentiation can be used as a means of checking the final answer
- After some practice, you may find Step 2 is not needed
    - Do use it on more awkward questions (negatives and fractions!)
- If the product **cannot** easily be identified, use **substitution**

Worked example

A curve has the gradient function $f'(x) = 5x^2\sin(2x^3)$.

Find an expression for $f(x)$.

$$f(x) = \int 5x^2 \sin(2x^3)\, dx$$

$$f(x) = 5\int x^2 \sin(2x^3)\, dx$$ Take 5 out as a factor

This is a product, almost in the form $g'(x)f(g(x))$

STEP 1: Spot the 'main' function

"the main function is sin (...) which would come from cos (...)"

STEP 2: 'Adjust and compensate' coefficients

"cos (...) would differentiate to −sin (...)"
"$2x^3$ would differentiate to $6x^2$"

$$f(x) = 5 \times - \times \frac{1}{6} \times \int - \times 6 \times x^2 \sin(2x^3)\, dx$$

compensate adjust

STEP 3: Integrate and simplify

$$\boxed{f(x) = -\frac{5}{6}\cos(2x^3) + c}$$

Substitution: Reverse Chain Rule

What is integration by substitution?

- When reverse chain rule is difficult to spot or awkward to use then **integration** by **substitution** can be used
  - **substitution** simplifies the integral by defining an alternative variable (usually u) in terms of the original variable (usually x)
  - **everything** (including "$\mathrm{d}x$" and **limits** for **definite integrals**) is then substituted which makes the integration much easier

How do I integrate using substitution?

STEP 1
Identify the substitution to be used – it will be the secondary function in the composite function

So $g(x)$ in $f(g(x))$ and $u = g(x)$

STEP 2
Differentiate the substitution and rearrange

$\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction

(i.e. "multiply by $\mathrm{d}x$" to get rid of fractions)

STEP 3
Replace all parts of the integral

All x terms should be replaced with equivalent u terms, including $\mathrm{d}x$

If finding a **definite integral** change the limits from x-values to u-values too

STEP 4
Integrate and either

substitute x back in

or

evaluate the definte integral using the u limits (either using a GDC or manually)

STEP 5

Find c, the constant of integration, if needed

- For **definite integrals**, a GDC should be able to process the integral without the need for a substitution
  - be clear about whether working is required or not in a question

Worked example

a) Find the integral

$$\int \frac{6x+5}{(3x^2+5x-1)^3}\,\mathrm{d}x$$

STEP 1: Identify the substitution

The composite function is $(3x^2+5x-1)^3$

The secondary function of this is $3x^2+5x-1$

$\therefore$ Let $u = 3x^2+5x-1$

STEP 2: Differentiate u and rearrange

$$\frac{du}{dx} = 6x+5$$

$$\therefore\ du = (6x+5)\,dx$$

STEP 3: Replace all parts of the integral

$$I = \int \frac{6x+5}{(3x^2+5x-1)^2}\,dx = \int \frac{du}{u^3}$$

$$= \int u^{-3}\,du$$

STEP 4: Integrate and substitute x back in

(STEP 5 not needed, evaluating c is not required)

$$I = \frac{u^{-2}}{-2} + c$$

$$I = -\frac{1}{2}(3x^2+5x-1)^{-2} + c$$

$$\boxed{\therefore\ I = \frac{-1}{2(3x^2+5x-1)^2} + c}$$

b) Evaluate the integral

$$\int_1^2 \frac{6x+5}{(3x^2+5x-1)^3}\,dx$$

giving your answer as an exact fraction in its simplest terms.

Note that you could use your GDC for this part
Certainly use it to check your answer!

From STEP 3 above, $I = \int_{x=1}^{x=2} u^{-3}\,du$

Change limits too, $x=1,\ u=3(1)^2+5(1)-1=7$

$x=2,\ u=3(2)^2+5(2)-1=21$

STEP 4: Integrate and evaluate

$I = \left[-\frac{1}{2}u^{-2}\right]_7^{21} = \left[-\frac{1}{2}(21)^{-2}\right]-\left[-\frac{1}{2}(7)^{-2}\right]$

$$\boxed{\therefore I = \frac{4}{441}}$$

5.4.3 Definite Integrals

Definite Integrals

What is a definite integral?

$$\int_a^b \mathrm{f}(x)\ dx = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- ***a*** and ***b*** are called limits
  - ***a*** is the lower limit
  - ***b*** is the upper limit
- $\mathrm{f}(x)$ is the **integrand**
- $\mathrm{F}(x)$ is an **antiderivative** of $\mathrm{f}(x)$
- The **constant** of **integration** (“**+c**”) is not needed in **definite integration**
  - “+c” would appear alongside both **F(*a*)** and **F(*b*)**
  - subtracting means the “+c”’s cancel

How do I find definite integrals analytically (manually)?

STEP 1
Give the integral a name to save having to rewrite the whole integral every time
If need be, rewrite the integral into an integrable form

$$I = \int_a^b \mathrm{f}(x)\ \mathrm{d}x$$

STEP 2
Integrate without applying the limits; you will not need “+c”
Notation: use square brackets [ ] with limits placed at the end bracket

STEP 3
Substitute the limits into the function and evaluate

Worked example

a) Show that

$$\int_2^4 3x(x^2-2)\,\mathrm{d}x = 144$$

STEP 1: Name the integral and rewrite into an integratable form

$$I = \int_2^4 (3x^3 - 6x)\,dx$$

STEP 2: Integrate

$$I = \left[\frac{3}{4}x^4 - 3x^2\right]_2^4$$

STEP 3: Evaluate

$$I = \left[\frac{3}{4}(4)^4 - 3(4)^2\right] - \left[\frac{3}{4}(2)^4 - 3(2)^2\right]$$

$$I = 144 - 0$$

$$\boxed{\therefore \int_2^4 3x(x^2-2) = 144}$$

b) Use your GDC to evaluate

$$\int_0^1 3e^{x^2 \sin x}\,\mathrm{d}x$$

giving your answer to three significant figures.

Using GDC,

$$\int_0^1 3e^{x^2 \sin x}\, dx = 3.872\,957\ldots$$

$$\boxed{\therefore \int_0^1 3e^{x^2 \sin x}\, dx = 3.87 \quad (3 \text{ s.f.})}$$

Properties of Definite Integrals

Fundamental Theorem of Calculus

$$\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$$

- Formally,
  - $f(x)$ is **continuous** in the interval $a \le x \le b$
  - $F(x)$ is an **antiderivative** of $f(x)$

What are the properties of definite integrals?

- Some of these have been encountered already and some may seem obvious ...
  - taking **constant** factors outside the integral
    - $\int_a^b k f(x)\ dx = k\int_a^b f(x)\ dx$ where k is a constant
    - useful when fractional and/or negative values involved
  - integrating term by term
    - $\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
    - the above works for subtraction of terms/functions too
  - equal upper and lower limits
    - $\int_a^a f(x)\ dx = 0$
    - on evaluating, this would be a value, subtract itself !
  - swapping limits gives the same, but negative, result
    - $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
    - compare 8 subtract 5 say, with 5 subtract 8 ...
  - splitting the interval
    - $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ where $a \le c \le b$
    - this is particularly useful for areas under multiple curves or areas under the x-axis
  - horizontal translations
    - $\int_a^b f(x)\ dx = \int_{a-k}^{b-k} f(x+k)\ dx$ where k is a constant
    - the graph of $y = f(x \pm k)$ is a horizontal translation of the graph of $y = f(x)$ ($f(x+k)$ translates left, $f(x-k)$ translates right)

Worked example

$\mathrm{f}(x)$ is a continuous function in the interval $5 \leq x \leq 15$.

It is known that $\int_5^{10} \mathrm{f}(x)\,\mathrm{d}x = 12$ and that $\int_{10}^{15} \mathrm{f}(x)\,\mathrm{d}x = 5$.

a) Write down the values of

i) $\int_7^7 \mathrm{f}(x)\,\mathrm{d}x$

ii) $\int_{10}^5 \mathrm{f}(x)\,\mathrm{d}x$

i. $\boxed{\int_7^7 f(x)\,dx = 0}$

"equal limits"

$\int_a^a f(x)\,dx = 0$

ii. $\boxed{\int_{10}^5 f(x)\,dx = -12}$

"swapped limits"

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

b) Find the values of

i) $\int_5^{15} \mathrm{f}(x)\,\mathrm{d}x$

ii) $\int_5^{10} 6\mathrm{f}(x+5)\,\mathrm{d}x$

i. $I=\int_5^{15} f(x)\,dx = \int_5^{10} f(x)\,dx + \int_{10}^{15} f(x)\,dx = 12+5=17$

"split limits"

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$$

$$\boxed{\therefore \int_5^{15} f(x)\,dx = 17}$$

ii. $I=\int_5^{10} 6f(x+5) = 6\int_5^{10} f(x+5)\,dx$

"factors"

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$I = 6\int_{5+5}^{10+5} f(x)\,dx$$

"horizontal translation"

$$\int_a^b f(x+k)\,dx = \int_{a+k}^{b+k} f(x)\,dx$$

$$I = 6\int_{10}^{15} f(x)\,dx = 6\times 5 = 30$$

$$\boxed{\therefore \int_5^{10} 6f(x+5)\,dx = 30}$$

5.4.4 Further Applications of Integration

Negative Integrals

- The area under a curve may appear **fully** or **partially** under the x-axis
  - This occurs when the function $f(x)$ takes **negative** values within the boundaries of the area
- The **definite integrals** used to find such **areas**
  - will be **negative** if the area is **fully** under the x-axis
  - possibly **negative** if the area is **partially** under the x-axis
    - this occurs if the negative area(s) is/are greater than the positive area(s), their **sum** will be **negative**
- When using a GDC use the modulus (absolute value) function so that all definite integrals have a positive value

$$A = \int_a^b |y| \, dx$$

  - This is given in the **formula booklet**

How do I find the area under a curve when the curve is fully under the x-axis?

STEP 1
Write the expression for the definite integral to find the area as usual

This may involve finding the lower and upper limits from a graph sketch or GDC and f(x) may need to be rewritten in an integrable form

STEP 2
The answer to the definite integral will be negative
Area must always be positive so take the modulus (absolute value) of it
e.g. If $I = -36$ then the area would be 36 (square units)

How do I find the area under a curve when the curve is partially under the x-axis?

- For questions that allow the use of a GDC you can still use

$$A = \int_a^c |f(x)| \, dx$$

- To find the area analytically (manually) use the following method

STEP 1
Split the area into parts - the area(s) that are above the x-axis and the area(s) that are below the x-axis

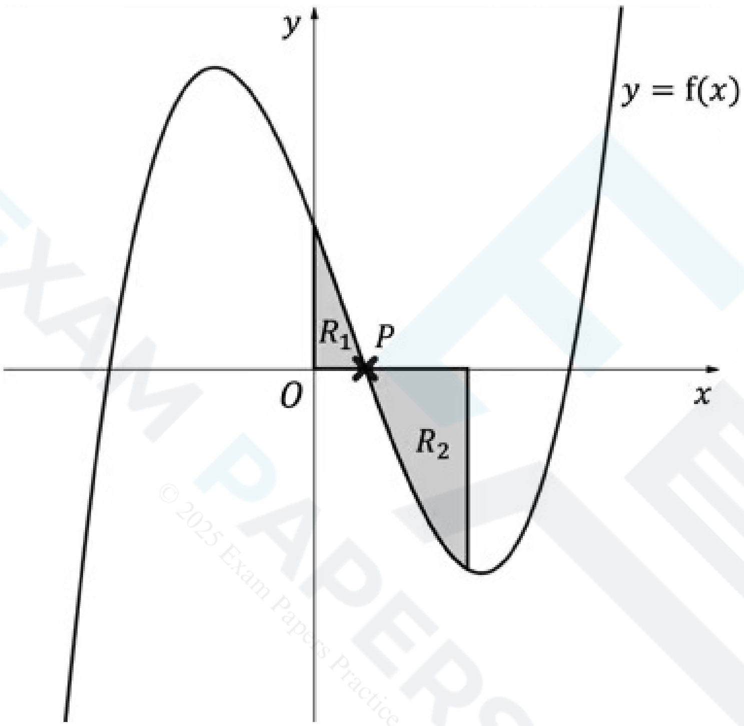
STEP 2
Write the expression for the definite integral for each part (give each part a name, I_1, I_2, etc)
This may involve finding the lower and upper limits of each part from a graph sketch or a GDC, finding the roots of the function (i.e. where $f(x) = 0$) and rewriting $f(x)$ in an integrable form

STEP 3
Find the value of each definite integral separately

STEP 4
Find the area by summing the modulus (absolute values) of each integral
(Mathematically this would be written $A = |I_1| + |I_2| + |I_3| + \ldots$)

Worked example

The diagram below shows the graph of $y = f(x)$ where $f(x) = (x+4)(x-1)(x-5)$.



The region R_1 is bounded by the curve $y = f(x)$, the x-axis and the y-axis.

The region R_2 is bounded by the curve $y = f(x)$, the x-axis and the line $x = 3$.

a) Determine the coordinates of the point labelled P.

a) The x-coordinate of P is a root of $f(x)$

$f(x) = 0$

$(x+4)(x-1)(x-5) = 0$

$x = -4, \; x = 1, \; x = 5$

Clearly from the graph, $x = 1$ at point P

$\boxed{\therefore P(1, 0)}$

b) i) Find a definite integral that would help find the area of the shaded region R_2 and briefly explain why this would **not** give the area of the region R_2.

ii) Find the exact area of the shaded region R_2.

b) i)

$$I_2 = \int_1^3 (x+4)(x-1)(x-5)\ dx$$

R_2 is underneath the x-axis so the value of the definite integral will be negative. Area cannot be negative.

ii) STEP 1:

$$I_2 = \int_1^3 (x+4)(x-1)(x-5)\ dx$$

$$I_2 = \int_1^3 (x^2+3x-4)(x-5)\ dx$$ Rewrite in an integrable form

$$I_2 = \int_1^3 (x^3-2x^2-19x+20)\ dx$$

$$I_2 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_1^3$$ Integrate (no need for "+c")

$$I_2 = \left(\frac{3^4}{4} - \frac{2(3)^3}{3} - \frac{19(3)^2}{2} + 20(3)\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{19}{2} + 20\right)$$

$$I_2 = -\frac{93}{4} - \frac{121}{12}$$

$$I_2 = -\frac{100}{3}$$

STEP 2: $\therefore$ Area of R_2, $A_2 = \frac{100}{3}$ square units

c) Find the exact total area of the shaded regions, R_1 and R_2.

c) STEP 1, 2: $A_1 = I_1 = \int_0^1 (x^3-2x^2-19x+20)\ dx$ Use the relevant results from b) ii)

$$I_1 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_0^1$$

STEP 3: $I_1 = \frac{121}{12} - 0$

STEP 4: $\therefore A_1 + A_2 = \frac{121}{12} + \frac{100}{3} = \frac{521}{12}$

$\therefore$ Total area shaded $= \frac{521}{12}$ square units

You can check the final answer using your GDC and the formula (in booklet) $A = \int_a^b |y|\ dx$.

Here, $A = \int_0^3 |(x+4)(x-1)(x-5)|\ dx$

$A = 43.41666...$

(Note that our GDC was not able to produce the exact answer...)

Area Between a Curve and a Line

- **Areas** whose boundaries include a **curve** and a (non-vertical) **straight line** can be found using integration
  - For an **area** under a **curve** a **definite integral** will be needed
  - For an **area** under a **line** the shape formed will be a **trapezium** or **triangle**
    - **basic area formulae** can be used rather than a definite integral
    - (although a definite integral would still work)
- The **area** required could be the **sum** or **difference** of areas under the curve and line

How do I find the area between a curve and a line?

STEP 1
If not given, sketch the graphs of the curve and line on the same diagram
Use a GDC to help with this step

STEP 2
Find the intersections of the curve and the line
If no diagram is given this will help identify the area(s) to be found

STEP 3
Determine whether the area required is the sum or difference of the area under the curve and the area under the line
Calculate the area under a curve using a integral of the form

$$\int_a^b y\,\mathrm{d}x$$

Calculate the area under a line using either $A = \frac{1}{2}bh$ for a triangle or $A = \frac{1}{2}h(a+b)$ for a trapezium (y-coordinates will be needed)

STEP 4
Evaluate the definite integrals and find their sum or difference as necessary to obtain the area required

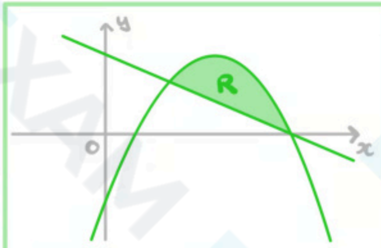
Worked example

The region R is bounded by the curve with equation $y = 10x - x^2 - 16$ and the line with equation $y = 8 - x$.

R lies entirely in the first quadrant.

a) Using your GDC, or otherwise, sketch the graphs of the curve and the line on the same diagram. Identify and label the region R on your sketch and use your GDC to find the x-coordinates of the points of intersection between the curve and the line.

STEP 1:



STEP 2: Points of intersection are $(3, 5)$ and $(8, 0)$

b) i) Write down an integral that would find the area of the region R.

ii) Find the area of the region R.

i) STEP 3: Curve is 'upper' boundary of R

$\therefore\ y_1 = 10x - x^2 - 16$

$y_2 = 8 - x$

$y_1 - y_2 = 10x - x^2 - 16 - (8 - x) = 11x - x^2 - 24$

$$\boxed{\therefore \text{ Area of } R,\ A_R = \int_3^8 (11x - x^2 - 24)\,dx}$$

ii) STEP 4:

$A_R = \int_3^8 (11x - x^2 - 24)\,dx$

$A_R = \left[\frac{11x^2}{2} - \frac{x^3}{3} - 24x\right]_3^8$

$A_R = \left[\frac{11(8)^2}{2} - \frac{(8)^3}{3} - 24(8)\right] - \left[\frac{11(3)^2}{2} - \frac{(3)^3}{3} - 24(3)\right]$

$A_R = -\frac{32}{3} - -\frac{63}{2}$

$$\boxed{\therefore \text{ Area of region } R \text{ is } \frac{125}{6} \text{ square units}}$$

Area Between 2 Curves

- **Areas** whose boundaries include **two curves** can be found by integration
  - The **area between two curves** will be the **difference** of the areas under the two curves
    - both areas will require a **definite integral**
  - Finding points of intersection may involve a more awkward equation than solving for a curve and a line

How do I find the area between two curves?

STEP 1
If not given, sketch the graphs of both curves on the same diagram
Use a GDC to help with this step

STEP 2
Find the intersections of the two curves
If no diagram is given this will help identify the area(s) to be found

STEP 3
For each area (there may only be one) determine which curve is the 'upper' boundary
For each area, write a definite integral of the form

$$\int_a^b (y_1 - y_2)\,\mathrm{d}x$$

where y_1 is the function for the 'upper' boundary and y_2 is the function for the 'lower' boundary
Be careful when there is more than one region – the 'upper' and 'lower' boundaries will swap

STEP 4
Evaluate the definite integrals and sum them up to find the total area
(Step 3 means no definite integral will have a negative value)

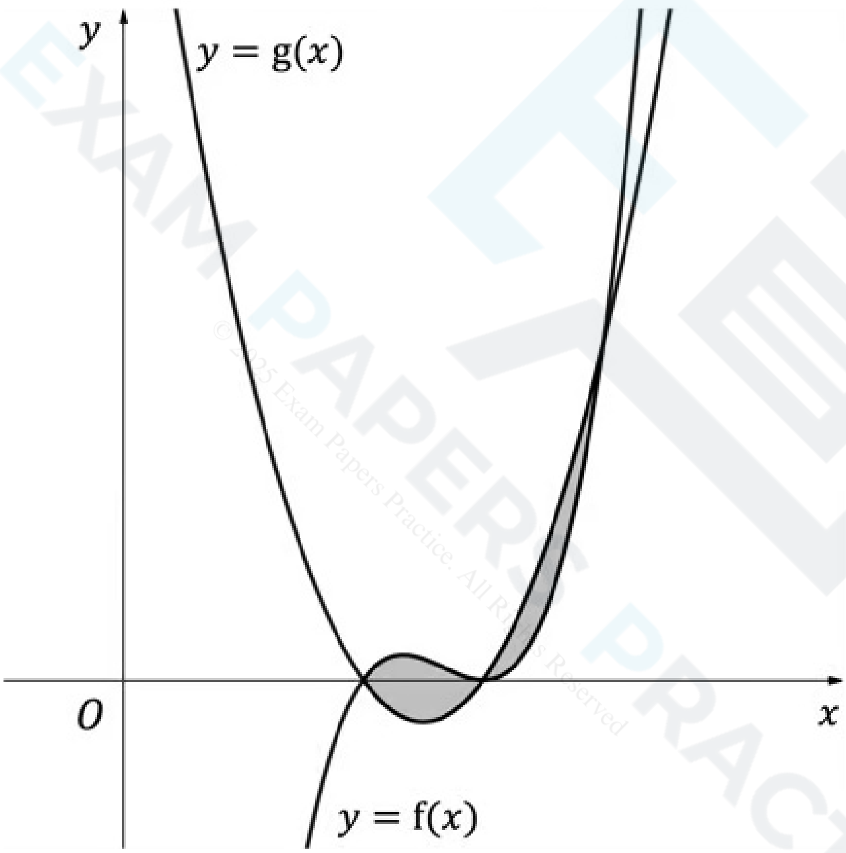
Worked example

The diagram below shows the curves with equations $y = \mathrm{f}(x)$ and $y = g(x)$ where

$\mathrm{f}(x) = (x-2)(x-3)^2$

$$\mathrm{g}(x) = x^2 - 5x + 6$$

Find the area of the shaded region.



STEP 1: Sketch of graph given

STEP 2: Two intersections are the roots of $f(x)$

$f(x) = (x-2)(x-3)^2 = 0$ at $x = 2$, $(y=0)$

and $x = 3$ $(y=0)$

Solve $f(x) = g(x)$ to find the other intersection

$(x-2)(x-3)^2 = x^2 - 5x + 6$

$(x-2)(x-3)^2 = (x-2)(x-3)$

$x - 3 = 1$

$x = 4$, $y = (4-2)(4-3) = 2$

STEP 3: The area, A_1 of the first region is given by

$A_1 = \int_2^3 \left[(x-2)(x-3)^2 - (x^2-5x+6)\right] dx$

$A_1 = \int_2^3 (x-2)(x-3)\left[(x-3) - 1\right] dx$ Factorise $(x-2)(x-3)$

$A_1 = \int_2^3 (x^2-5x+6)(x-4)\, dx$

$A_1 = \int_2^3 (x^3 - 9x^2 + 26x - 24)\, dx$

$A_1 = \left[\frac{x^4}{4} - 3x^3 + 13x^2 - 24x\right]_2^3$

$A_1 = \left(\frac{(3)^4}{4} - 3(3)^3 + 13(3)^2 - 24(3)\right) - \left(\frac{(2)^4}{4} - 3(2)^3 + 13(2)^2 - 24(2)\right)$

$A_1 = -\frac{63}{4} - -16 = \frac{1}{4}$

For A_2, the 'upper' and 'lower' boundaries swap

$A_2 = \int_3^4 \left[(x^2-5x+6) - (x-2)(x-3)^2\right] dx$

$A_2 = \int_3^4 (x-2)(x-3)\left[1 - (x-3)\right] dx$

$A_2 = \int_3^4 (x^2-5x+6)(4-x)\, dx$

$A_2 = \int_3^4 (-x^3 + 9x^2 - 26x + 24)\, dx$

$A_2 = \left[-\frac{x^4}{4} + 3x^3 - 13x^2 + 24x\right]_3^4$

$A_2 = \left(-\frac{(4)^4}{4} + 3(4)^3 - 13(4)^2 + 24(4)\right) - \left(-\frac{(3)^4}{4} + 3(3)^3 - 13(3)^2 + 24(3)\right)$

$A_2 = 16 - \frac{63}{4} = \frac{1}{4}$

Total area is $A_1 + A_2$

$\boxed{\therefore \text{ Area of shaded region is } \frac{1}{2} \text{ square unit}}$