



DP IB Maths: AI HL

5.3 Integration

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5.3.1 Trapezoid Rule: Numerical Integration

Trapezoid Rule: Numerical Integration

What is the trapezoid rule?

- The **trapezoidal rule** is a numerical method used to find the **approximate area** enclosed by a curve, the X -axis and two vertical lines
 - it is also known as '**trapezoid rule**' and '**trapezium rule**'
- The trapezoidal rule finds an **approximation** of the area by **summing of the areas** of trapezoids beneath the curve
 - $y_0 = f(a)$, $y_1 = f(a + h)$, $y_2 = f(a + 2h)$ etc

$$\int_a^b f(x) dx \approx \frac{1}{2}h \left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1}) \right]$$

$$\text{where } h = \frac{b - a}{n}$$

- Note that there are n trapezoids (also called strips) but $(n + 1)$ function values (y_i)
- The trapezoidal rule is given in the **formula booklet**

What else can I be asked to do with the trapezoid rule?

- Comparing the **true** answer with the answer from the trapezoid rule
 - This may involve finding the **percentage error** in the approximation
 - The true answer may be given in the question, found from a GDC or from work on **integration**

Worked example

- a) Using the trapezoidal rule, find an approximate value for

$$\int_0^4 \frac{6x^2}{x^3 + 2} dx$$

to 3 decimal places, using $n = 4$.

a) $h = \frac{4-0}{4} = 1$

STEP 1: FIND $h = \frac{b-a}{n}$
 USING \int_a^b AND $n=4$

x	$y = \frac{6x^2}{x^3 + 2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.862...$
$x_4 = 4$	$y_4 = 1.454...$

STEP 2: USING WIDTH $h=1$, VALUES ARE 0, 1, 2, 3, 4

STEP 3: SETUP TABLE AND USE EQUATION TO CALCULATE y VALUES

STEP 4: USING FORMULA FROM FORMULA BOOKLET

$$\int_a^b y dx \approx \frac{1}{2} h (y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1}))$$

SUBSTITUTE ALL VALUES

$$\begin{aligned}
 & \frac{1}{2} \times 1 \times (0 + 1.454... + 2(2 + 2.4 + 1.862...)) \\
 &= \frac{1}{2} (1.454... + 12.524...) \\
 &= 6.9893... = 6.989 \text{ (3 dp)}
 \end{aligned}$$

- b) Given that the area bounded by the curve, the X -axis and the lines $X = 0$ and $X = 4$ is 6.993 to three decimal places, calculate the percentage error in the trapezoidal rule approximation.

b)
$$\frac{6.989 - 6.993}{6.993} \times 100$$

$$= -0.\dot{0}5720\dot{0}$$

$$= 0.06\% \text{ (2 dp)}$$

USING
% ERROR = $\frac{\text{ESTIMATE} - \text{EXACT}}{\text{EXACT}} \times 100$

IGNORE MINUS, YOU COULD
DO EXACT-ESTIMATE TO
GET RID OFF THIS

5.3.2 Introduction to Integration

Introduction to Integration

What is integration?

- **Integration** is the opposite to **differentiation**
 - Integration is referred to as **antidifferentiation**
 - The result of integration is referred to as the **antiderivative**
- **Integration** is the process of finding the expression of a function (**antiderivative**) from an expression of the **derivative** (**gradient function**)

What is the notation for integration?

- An **integral** is normally written in the form

$$\int f(x) \, dx$$

- the large operator \int means “integrate”
- “**dx**” indicates which variable to integrate with respect to
- $f(x)$ is the function to be integrated (sometimes called the integrand)
- The **antiderivative** is sometimes denoted by $F(x)$
 - there's then no need to keep writing the whole integral; refer to it as $F(x)$
- $F(x)$ may also be called the **indefinite integral** of $f(x)$

What is the constant of integration?

- Recall one of the special cases from **Differentiating Powers of x**
 - If $f(x) = a$ then $f'(x) = 0$
- This means that integrating 0 will produce a **constant** term in the antiderivative
 - a zero term wouldn't be written as part of a function
 - **every** function, when integrated, potentially has a **constant** term
- This is called the **constant of integration** and is usually denoted by the letter **C**
 - it is often referred to as “plus **C**”
- Without more information it is impossible to deduce the value of this constant
 - there are endless antiderivatives, $F(x)$, for a function $f(x)$

Integrating Powers of x

How do I integrate powers of x?

- Powers of x are integrated according to the following formulae:
 - If $f(x) = x^n$ then $\int f(x) \, dx = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and c is the **constant of integration**
 - This is given in the **formula booklet**
- If the power of x is multiplied by a constant then the integral is also multiplied by that constant
 - If $f(x) = ax^n$ then $\int f(x) \, dx = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and a is a constant and c is the **constant of integration**
- $\frac{dy}{dx}$ notation can still be used with integration
- Note that the formulae above do not apply when $n = -1$ as this would lead to division by zero
- Remember the special case:
 - $\int a \, dx = ax + c$
 - e.g. $\int 4 \, dx = 4x + c$
 - This allows **constant** terms to be integrated
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
 - e.g. If $f(x) = 5\sqrt[3]{x}$ then rewrite as $f(x) = 5x^{\frac{1}{3}}$ and integrate
- Functions involving **fractions** with **denominators** in **terms** of x will need to be rewritten as **negative powers** of x first
 - e.g. If $f(x) = \frac{4}{x^2} + x^2$ then rewrite as $f(x) = 4x^{-2} + x^2$ and integrate
- The formulae for integrating powers of x apply to **all rational numbers** so it is possible to integrate any expression that is a sum or difference of powers of x
 - e.g. If $f(x) = 8x^3 - 2x + 4$ then

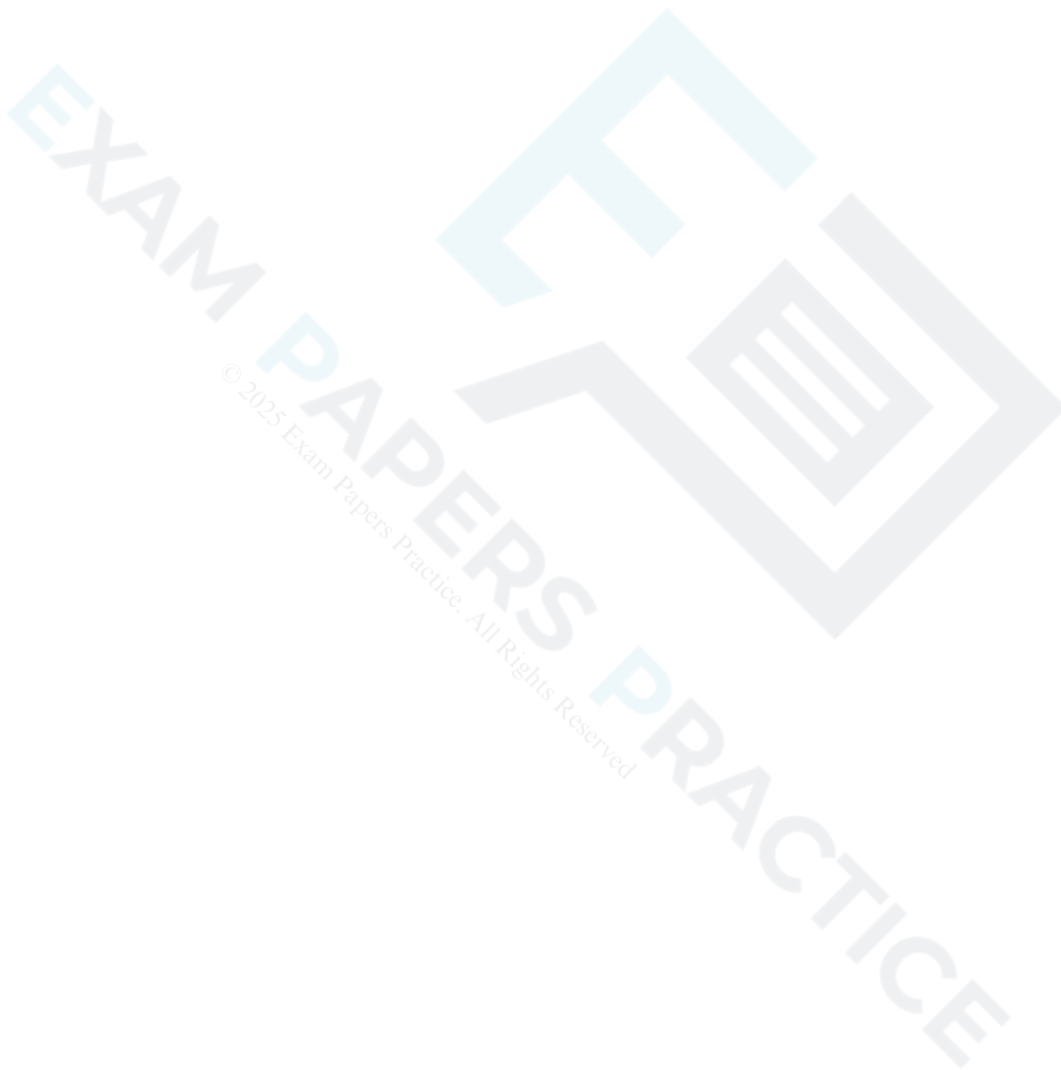
$$\int f(x) \, dx = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$$
- **Products** and **quotients** cannot be integrated this way so would need **expanding/simplifying** first

- e.g. If $f(x) = 8x^2(2x - 3)$ then

$$\int f(x) \, dx = \int (16x^3 - 24x^2) \, dx = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$$

What might I be asked to do once I've found the anti-derivative (integrated)?

- With more information the **constant of integration**, C , can be found
- The **area under a curve** can be found using integration



Worked example

Given that

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - \frac{1}{\sqrt{x}}$$

find an expression for y in terms of x .

Firstly rewrite all terms as powers of x

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}} \leftarrow \text{fractional AND negative!}$$

$$y = \int (3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}) dx$$

$$\therefore y = \frac{3x^5}{5} - \frac{2x^3}{3} + 3x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

special case \uparrow constant of integration \uparrow
 take care with negatives, $-\frac{1}{2} + 1 = \frac{1}{2}$

$$\therefore y = \frac{3}{5}x^5 - \frac{2}{3}x^3 + 3x - 2\sqrt{x} + c$$

5.3.3 Applications of Integration

Finding the Constant of Integration

What is the constant of integration?

- When finding an **anti-derivative** there is a constant term to consider
 - this constant term, usually called **C**, is the **constant of integration**
- In terms of **graphing** an **anti-derivative**, there are endless possibilities
 - collectively these may be referred to as the **family of antiderivatives** or **family of curves**
 - the constant of integration is determined by the **exact** location of the curve
 - if a **point** on the **curve** is **known**, the **constant of integration** can be found

How do I find the constant of integration?

- For $F(x) + c = \int f(x) dx$, the **constant of integration**, **C** – and so the particular **antiderivative** – can be found if a point the graph of $y = F(x) + c$ passes through is known

STEP 1

If need be, rewrite $f(x)$ into an integrable form

Each term needs to be a power of **x** (or a constant)

STEP 2

Integrate each term of $f(x)$, remembering the constant of integration, “+ c”

(Increase power by 1 and divide by new power)

STEP 3

Substitute the **x** and **y** coordinates of a given point in to $F(x) + c$ to form an equation in **c**

Solve the equation to find **c**

Worked example

The graph of $y = f(x)$ passes through the point $(3, -4)$. The gradient function of $f(x)$ is given by $f'(x) = 3x^2 - 4x - 4$.

Find $f(x)$.

STEP 1 $f'(x)$ is already in an integrable form

$$f'(x) = 3x^2 - 4x - 4$$

STEP 2 Integrate, remembering "+c"

$$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} - 4x + c$$

$$f(x) = x^3 - 2x^2 - 4x + c$$

STEP 3 Substitute x and y coordinates to find c

$$f(3) = -4$$

$$\therefore (3)^3 - 2(3)^2 - 4(3) + c = -4$$

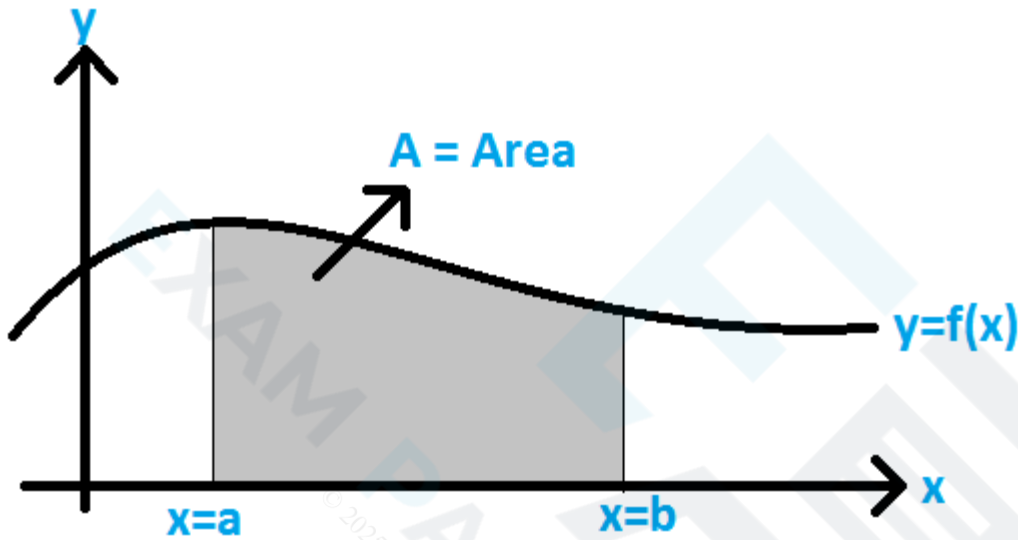
$$27 - 18 - 12 + c = -4$$

$$c = -1$$

$$\therefore f(x) = x^3 - 2x^2 - 4x - 1$$

Area Under a Curve Basics

What is meant by the area under a curve?



- The phrase "**area under a curve**" refers to the area bounded by
 - the graph of $y = f(x)$
 - the x -axis
 - the **vertical** line $x = a$
 - the **vertical** line $x = b$
- The **exact area under a curve** is found by evaluating a **definite integral**
- The graph of $y = f(x)$ could be a **straight line**
 - the use of **integration** described below would still apply
 - but the shape created would be a **trapezoid**
 - so it is easier to use " $A = \frac{1}{2}h(a + b)$ "

What is a definite integral?

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- **a** and **b** are called limits
 - **a** is the **lower** limit
 - **b** is the **upper** limit
- $f(x)$ is the **integrand**
- $F(x)$ is an **antiderivative** of $f(x)$
- The **constant of integration** (" $+ C$ ") is not needed in **definite integration**
 - " $+ C$ " would appear alongside both **F(a)** and **F(b)**
 - subtracting means the " $+ C$ "'s cancel

How do I form a definite integral to find the area under a curve?

- The graph of $y = f(x)$ and the x -axis should be obvious boundaries for the area so the key here is in finding **a** and **b** - the **lower** and **upper** limits of the **integral**

STEP 1

Use the given sketch to help locate the limits

You may prefer to plot the graph on your GDC and find the limits from there

STEP 2

Look carefully where the 'left' and 'right' boundaries of the area lie

If the boundaries are vertical lines, the limits will come directly from their equations

Look out for the y -axis being one of the (vertical) boundaries - in this case the limit (x) will be 0

One, or both, of the limits, could be a root of the equation $f(x) = 0$

i.e. where the graph of $y = f(x)$ crosses the x -axis

In this case solve the equation $f(x) = 0$ to find the limit(s)

A GDC will solve this equation, either from the graphing screen or the equation solver

STEP 3

The definite integral for finding the area can now be set up in the form

$$A = \int_a^b f(x) \, dx$$

Definite Integrals using GDC

Does my calculator/GDC do definite integrals?

- Modern graphic calculators (and some 'advanced' scientific calculators) have the functionality to evaluate **definite integrals**
 - i.e. they can calculate the **area under a curve** (see above)
- If a calculator has a button for evaluating definite integrals it will look something like

$$\int_{\square}^{\square} \square$$

- This may be a physical button or accessed via an on-screen menu
- Some GDCs may have the ability to find the area under a curve from the graphing screen
- Be careful with **any** calculator/GDC, they may not produce an **exact** answer

How do I use my GDC to find definite integrals?

Without graphing first ...

- Once you know the **definite integral** function your calculator will need three things in order to evaluate it
 - The function to be integrated (**integrand**) ($f(x)$)
 - The **lower** limit (a from $x = a$)
 - The **upper** limit (b from $x = b$)
- Have a play with the order in which your calculator expects these to be entered – some do not always work left to right as it appears on screen!

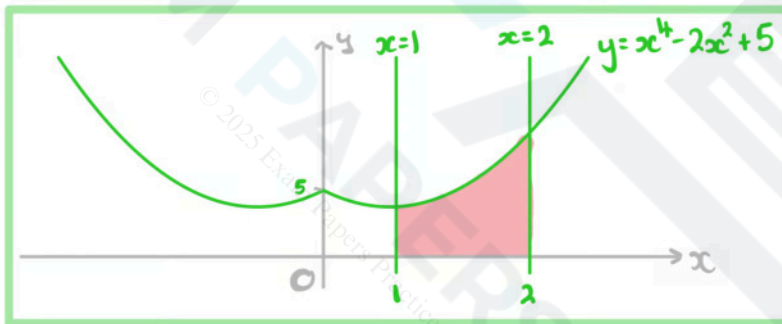
With graphing first ...

- Plot the graph of $y = f(x)$
 - You may also wish to plot the vertical lines $x = a$ and $x = b$
 - make sure your GDC is expecting an " $x =$ " style equation
 - Once you have plotted the graph you need to look for an option regarding "area" or a physical button
 - it may appear as the integral symbol (e.g. $\int dx$)
 - your GDC may allow you to select the lower and upper limits by moving a cursor along the curve – however this may not be very accurate
 - your GDC may allow you to type the exact limits required from the keypad
 - the lower limit would be typed in first
 - read any information that appears on screen carefully to make sure

Worked example

- a) Using your GDC to help, or otherwise, sketch the graphs of
 $y = x^4 - 2x^2 + 5$,
 $x = 1$ and
 $x = 2$ on the same diagram

Use the 'graph' menu on your GDC to plot $y = x^4 - 2x^2 + 5$.
 You may then need to change the 'input type' to 'x='
 to enter $x = 1$ and $x = 2$.
 Plot the graph on your GDC and sketch the result, ensuring
 to include all the main properties of each graph.



- b) The area enclosed by the three graphs from part (a) and the x -axis is to be found.
 Write down an integral that would find this area.

$$\int_1^2 (x^4 - 2x^2 + 5) \, dx$$

- c) Using your GDC, or otherwise, find the exact area described in part (b).
 Give your answer in the form $\frac{a}{b}$ where a and b are integers.

$$\text{Area} = \int_1^2 (x^4 - 2x^2 + 5) \, dx = \frac{98}{15} \text{ square units}$$

From the graphing screen on our GDC the integral value was given as 6.53333333 - not exact!