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5.3 Integration

IB Maths - Revision Notes

5.3.1 Trapezoid Rule : Numerical Integration

Trapezoid Rule : Numerical Integration

What is the trapezoid rule?

- The **trapezoidal rule** is a numerical method used to find the **approximate area** enclosed by a curve, the x-axis and two vertical lines
  - it is also known as '**trapezoid rule**' and '**trapezium rule**'
- The trapezoidal rule finds an **approximation** of the area by **summing of the areas** of trapezoids beneath the curve
  - $y_0 = f(a), \quad y_1 = f(a+h), \quad y_2 = f(a+2h)$ etc

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right]$$

where $h = \dfrac{b-a}{n}$

- Note that there are n trapezoids (also called strips) but $(n+1)$ function values (y_i)
- The trapezoidal rule is given in the **formula booklet**

What else can I be asked to do with the trapezoid rule?

- Comparing the **true** answer with the answer from the trapezoid rule
  - This may involve finding the **percentage error** in the approximation
  - The true answer may be given in the question, found from a GDC or from work on **integration**

Exam Tip

- Ensure you are clear about the difference between the number of data points (y values) and the number of strips (number of trapezoids) used in a Trapezoid Rule question
- Although it shouldn't be too much trouble to type the trapezoid rule into your GDC in one go, it may be wise to work parts of it out separately and write these down as part of your working out

Worked example

a) Using the trapezoidal rule, find an approximate value for

$$\int_0^4 \frac{6x^2}{x^3+2}\,dx$$

to 3 decimal places, using $n = 4$.

a) $h = \frac{4-0}{4} = 1$

STEP 1: FIND $h = \frac{b-a}{n}$ USING $\int_{a=0}^{b=4}$ AND $n = 4$

STEP 2: USING WIDTH $h = 1$, VALUES ARE 0, 1, 2, 3, 4

STEP 3: SETUP TABLE AND USE EQUATION TO CALCULATE y VALUES

x	$y = \frac{6x^2}{x^3+2}$
$x_0 = 0$	$y_0 = 0$
$x_1 = 1$	$y_1 = 2$
$x_2 = 2$	$y_2 = 2.4$
$x_3 = 3$	$y_3 = 1.862...$
$x_4 = 4$	$y_4 = 1.454...$

STEP 4: USING FORMULA FROM FORMULA BOOKLET

$$\int_a^b y\,dx \approx \frac{1}{2}h\left(y_0 + y_n + 2(y_1 + y_2 + ... + y_{n-1})\right)$$

SUBSTITUTE ALL VALUES

$$\frac{1}{2} \times 1 \times (0 + 1.454... + 2(2 + 2.4 + 1.862...))$$

$$= \frac{1}{2}(1.454... + 12.524...)$$

$$= 6.9893... = 6.989 \quad (3 \text{ dp})$$

b) Given that the area bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is 6.993 to three decimal places, calculate the percentage error in the trapezoidal rule approximation.

USING $\%\ \text{ERROR} = \frac{\text{ESTIMATE} - \text{EXACT}}{\text{EXACT}} \times 100$

b) $$\frac{6.989 - 6.993}{6.993} \times 100$$

$$= -0.\dot{0}5720\dot{0}$$

IGNORE MINUS, YOU COULD DO EXACT−ESTIMATE TO GET RID OFF THIS

$$= 0.06\% \quad (2 \text{ dp})$$

5.3.2 Introduction to Integration

Introduction to Integration

What is integration?

- **Integration** is the opposite to **differentiation**
  - Integration is referred to as **antidifferentiation**
  - The result of integration is referred to as the **antiderivative**
- **Integration** is the process of finding the expression of a function (**antiderivative**) from an expression of the **derivative** (**gradient function**)

What is the notation for integration?

- An **integral** is normally written in the form

$$\int f(x)\ \mathrm{d}x$$

  - the large operator $\int$ means "integrate"
  - "$\mathrm{d}x$" indicates which variable to integrate with respect to
  - $f(x)$ is the function to be integrated (sometimes called the integrand)
- The **antiderivative** is sometimes denoted by $\mathrm{F}(x)$
  - there's then no need to keep writing the whole integral; refer to it as $\mathrm{F}(x)$
- $\mathrm{F}(x)$ may also be called the **indefinite integral** of $f(x)$

What is the constant of integration?

- Recall one of the special cases from **Differentiating Powers of x**
  - If $f(x) = a$ then $f'(x) = 0$
- This means that integrating 0 will produce a **constant** term in the antiderivative
  - a zero term wouldn't be written as part of a function
  - **every** function, when integrated, potentially has a **constant** term
- This is called the **constant** of **integration** and is usually denoted by the letter c
  - it is often referred to as "plus c"
- Without more information it is impossible to deduce the value of this constant
  - there are endless antiderivatives, $\mathrm{F}(x)$, for a function $f(x)$

Integrating Powers of x

How do I integrate powers of x?

- Powers of x are integrated according to the following formulae:
  - If $f(x) = x^n$ then $\int f(x)\ \mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}, n \neq -1$ and c is the **constant of integration**
  - This is given in the **formula booklet**
- If the power of x is multiplied by a constant then the integral is also multiplied by that constant
  - If $f(x) = ax^n$ then $\int f(x)\ \mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}, n \neq -1$ and a is a constant and c is the **constant** of **integration**
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation can still be used with integration
- Note that the formulae above do not apply when $x = -1$ as this would lead to division by zero
- Remember the special case:
  - $\int a\ \mathrm{d}x = ax + c$
    - e.g. $\int 4\ \mathrm{d}x = 4x + c$
  - This allows **constant** terms to be integrated
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
  - eg. If $f(x) = 5\sqrt[3]{x}$ then rewrite as $f(x) = 5x^{\frac{1}{3}}$ and integrate
- Functions involving **fractions** with **denominators** in **terms** of x will need to be rewritten as **negative powers** of x first
  - e.g. If $f(x) = \frac{4}{x^2} + x^2$ then rewrite as $f(x) = 4x^{-2} + x^2$ and integrate
- The formulae for integrating powers of x apply to **all rational numbers** so it is possible to integrate any expression that is a sum or difference of powers of x
  - e.g. If $f(x) = 8x^3 - 2x + 4$ then

$$\int f(x)\ \mathrm{d}x = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$$

- **Products** and **quotients** cannot be integrated this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = 8x^2(2x-3)$ then

$$\int f(x)\ \mathrm{d}x = \int (16x^3 - 24x^2)\ \mathrm{d}x = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$$

What might I be asked to do once I've found the anti-derivative (integrated)?

- With more information the **constant of integration**, c, can be found
- The **area under** a **curve** can be found using integration

Exam Tip

- You can speed up the process of integration in the exam by committing the pattern of basic integration to memory
  - In general you can think of it as 'raising the power by one and dividing by the new power'
  - Practice this lots before your exam so that it comes quickly and naturally when doing more complicated integration questions

Worked example

Given that

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - \frac{1}{\sqrt{x}}$$

find an expression for y in terms of x.

Firstly rewrite all terms as powers of x

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}$$ ← fractional AND negative!

$$y = \int \left(3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}\right) dx$$

$$\therefore y = \frac{3x^5}{5} - \frac{2x^3}{3} + 3x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Special case

take care with negatives, $-\frac{1}{2} + 1 = \frac{1}{2}$

constant of integration

$$\boxed{\therefore y = \frac{3}{5}x^5 - \frac{2}{3}x^3 + 3x - 2\sqrt{x} + c}$$

5.3.3 Applications of Integration

Finding the Constant of Integration

What is the constant of integration?

- When finding an **anti-derivative** there is a constant term to consider
  - this constant term, usually called c, is the **constant** of **integration**
- In terms of **graphing** an **anti-derivative**, there are endless possibilities
  - collectively these may be referred to as the **family** of **antiderivatives** or **family** of **curves**
  - the constant of integration is determined by the **exact** location of the curve
    - if a **point** on the **curve** is **known,** the **constant** of **integration** can be found

How do I find the constant of integration?

- For $\mathrm{F}(x)+c=\int f(x)\,\mathrm{d}x$, the **constant** of **integration, c** – and so the particular **antiderivative** – can be found if a point the graph of $y=\mathrm{F}(x)+c$ passes through is known

STEP 1
If need be, rewrite $f(x)$ into an integrable form
Each term needs to be a power of x (or a constant)

STEP 2
Integrate each term of $f'(x)$, remembering the constant of integration, "$+c$"
(Increase power by 1 and divide by new power)

STEP 3
Substitute the x and y coordinates of a given point in to $\mathrm{F}(x)+c$ to form an equation in c
Solve the equation to find c

> **Exam Tip**
>
> - If a constant of integration can be found then the question will need to give you some extra information
>   - If this is given then make sure you use it to find the value of c

Worked example

The graph of $y = f(x)$ passes through the point $(3, -4)$. The gradient function of $f(x)$ is given by $f'(x) = 3x^2 - 4x - 4$.

Find $f(x)$.

STEP 1 $f'(x)$ is already in an integrable form

$f'(x) = 3x^2 - 4x - 4$

STEP 2 Integrate, remembering "+c"

$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} - 4x + c$

$f(x) = x^3 - 2x^2 - 4x + c$

STEP 3 Substitute x and y coordinates to find c

$f(3) = -4$

$\therefore (3)^3 - 2(3)^2 - 4(3) + c = -4$

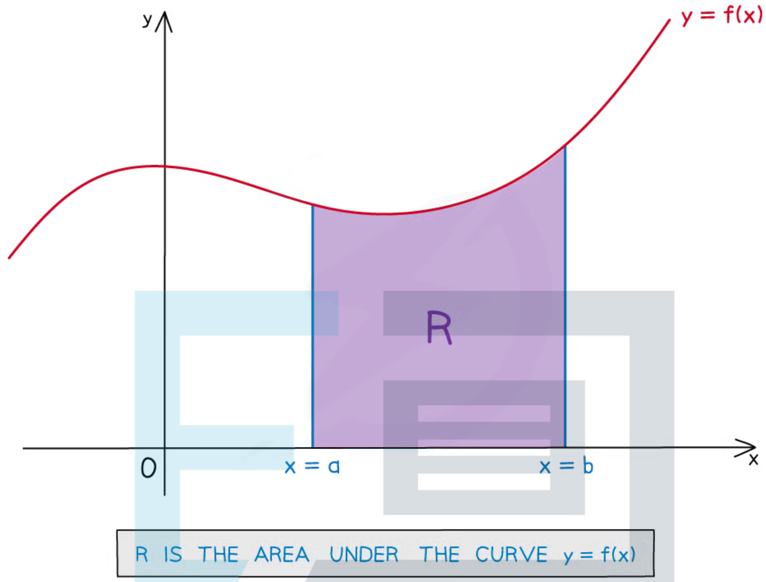
$27 - 18 - 12 + c = -4$

$c = -1$

$\boxed{\therefore f(x) = x^3 - 2x^2 - 4x - 1}$

Area Under a Curve Basics

What is meant by the area under a curve?



- The phrase "**area under a curve**" refers to the area bounded by
  - the graph of $y = f(x)$
  - the x-axis
  - the **vertical** line $x = a$
  - the **vertical** line $x = b$
- The **exact area under a curve** is found by evaluating a **definite integral**
- The graph of $y = f(x)$ could be a **straight line**
  - the use of **integration** described below would still apply
    - but the shape created would be a **trapezoid**
    - so it is easier to use "$A = \frac{1}{2}h(a+b)$"

What is a definite integral?

$$\int_a^b f(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- **a** and **b** are called limits
  - **a** is the **lower** limit
  - **b** is the **upper** limit
- $f(x)$ is the **integrand**

- $F(x)$ is an **antiderivative** of $f(x)$
- The **constant of integration** ("$+c$") is not needed in **definite integration**
  - "$+c$" would appear alongside both **F(a)** and **F(b)**
  - subtracting means the "$+c$"'s cancel

How do I form a definite integral to find the area under a curve?

- The graph of $y = f(x)$ and the x-axis should be obvious boundaries for the area so the key here is in finding a and b - the **lower** and **upper** limits of the **integral**

STEP 1

Use the given sketch to help locate the limits

You may prefer to plot the graph on your GDC and find the limits from there

STEP 2

Look carefully where the 'left' and 'right' boundaries of the area lie

If the boundaries are vertical lines, the limits will come directly from their equations

Look out for the y-axis being one of the (vertical) boudnaries – in this case the limit (x) will be 0

One, or both, of the limits, could be a root of the equation $f(x) = 0$

i.e. where the graph of $y = f(x)$ crosses the x-axis

In this case solve the equation $f(x) = 0$ to find the limit(s)

A GDC will solve this equation, either from the graphing screen or the equation solver

STEP 3

The definite integral for finding the area can now be set up in the form

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Exam Tip

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed)
- Add information to any diagram provided in the question, as well as axes intercepts and values of limits
  - Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

Definite Integrals using GDC

Does my calculator/GDC do definite integrals?

- Modern graphic calculators (and some 'advanced' scientific calculators) have the functionality to evaluate **definite integrals**
  - i.e. they can calculate the **area under a curve** (see above)
- If a calculator has a button for evalutaing definite integrals it will look something like

$$\int_{\square}^{\square} \square$$

- This may be a physical button or accessed via an on-screen menu
- Some GDCs may have the ability to find the area under a curve from the graphing screen
- Be careful with **any** calculator/GDC, they may not produce an **exact** answer

How do I use my GDC to find definite integrals?

Without graphing first ...

- Once you know the **definite integral** function your calculator will need three things in order to evaluate it
  - The function to be integrated (**integrand**) ($f(x)$)
  - The **lower** limit (a from $x = a$)
  - The **upper** limit (b from $x = b$)
- Have a play with the order in which your calculator expects these to be entered – some do not always work left to right as it appears on screen!

With graphing first ...

- Plot the graph of $y = f(x)$
  - You may also wish to plot the vertical lines $x = a$ and $x = b$
    - make sure your GDC is expecting an "$x =$" style equation
  - Once you have plotted the graph you need to look for an option regarding "area" or a physical button
    - it may appear as the integral symbol (e.g. $\int dx$)
    - your GDC may allow you to select the lower and upper limits by moving a cursor along the curve - however this may not be very accurate
    - your GDC may allow you to type the exact limits required from the keypad
      - the lower limit would be typed in first
      - read any information that appears on screen carefully to make sure

> **Exam Tip**
>
> - When revising for your exams always use your GDC to check any definite integrals you have carried out by hand
>   - This will ensure you are confident using the calculator you plan to take into the exam and should also get you into the habit of using you GDC to check your work, something you should do if possible

Worked example

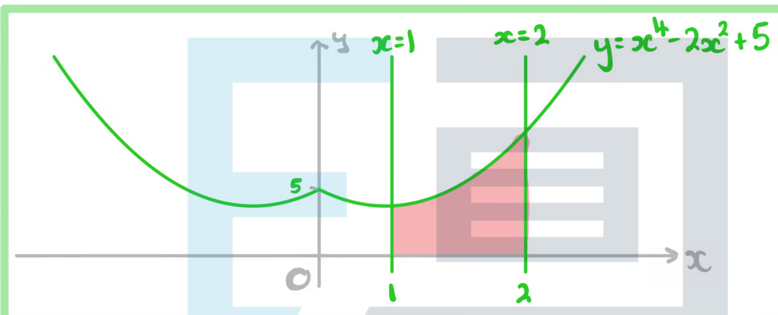
a) Using your GDC to help, or otherwise, sketch the graphs of

$y = x^4 - 2x^2 + 5$,

$x = 1$ and

$x = 2$ on the same diagram

Use the 'graph' menu on your GDC to plot $y = x^4 - 2x^2 + 5$.
You may then need to change the 'input type' to '$x =$'
to enter $x = 1$ and $x = 2$.
Plot the graph on your GDC and sketch the result, ensuring
to include all the main properties of each graph.



b) The area enclosed by the three graphs from part (a) and the x-axis is to be found.
Write down an integral that would find this area.

$$\int_1^2 (x^4 - 2x^2 + 5)\, dx$$

c) Using your GDC, or otherwise, find the exact area described in part (b).

Give your answer in the form $\frac{a}{b}$ where a and b are integers.

$$\text{Area} = \int_1^2 (x^4 - 2x^2 + 5)\, dx = \frac{98}{15} \text{ square units}$$

From the graphing screen on our GDC the integral value was given as 6.53333333 – not exact!