5.2 Further Differentiation

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Course	DP IB Maths
Section	5. Calculus
Topic	5.2 Further Differentiation
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Quotient Rule

Let $u = 5x^7$ $\quad v = \sin 2x$

$\frac{du}{dx} = 35x^6$ $\quad \frac{dv}{dx} = 2\cos 2x$ chain rule

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{(\sin 2x)35x^6 - 5x^7(2\cos 2x)}{(\sin 2x)^2}$$

$$\boxed{\frac{d\left(\frac{5x^7}{\sin 2x}\right)}{dx} = \frac{35x^6\sin 2x - 10x^7\cos 2x}{\sin^2 2x}}$$

Question 2

a) Chain rule ①

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let $y = \cos u$ $\quad u = x^2 - 3x + 7$

$\frac{dy}{du} = -\sin u$ $\quad \frac{du}{dx} = 2x - 3$

$\frac{dy}{dx} = (-\sin u)(2x-3)$

sub in $u = x^2 - 3x + 7$

$\frac{dy}{dx} = (-\sin(x^2-3x+7))(2x-3)$

Chain rule ②

Let $y = \sin u$ $\quad u = e^x$

$\frac{dy}{du} = \cos u$ $\quad \frac{du}{dx} = e^x$

$\frac{dy}{dx} = (\cos u)e^x$

sub in $u = e^x$, $\frac{dy}{dx} = (\cos(e^x))e^x$

$$\boxed{\frac{dy}{dx} = -(2x-3)\sin(x^2-3x+7) + e^x\cos(e^x)}$$

b) Method 1: Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$y = \ln u$ $\qquad$ $u = 2x^3$

$\frac{dy}{du} = \frac{1}{u}$ $\qquad$ $\frac{du}{dx} = 6x^2$

$$\frac{dy}{dx} = \frac{1}{u}(6x^2)$$

sub $u = 2x^3$

$$\frac{dy}{dx} = \frac{1}{2x^3}(6x^2) = \boxed{\frac{3}{x}}$$

OR

Method 2: Simplify using log laws

$$y = \ln(2x^3) = \ln 2 + \ln x^3 = \ln 2 + 3\ln x$$

$$\frac{dy}{dx} = \boxed{\frac{3}{x}}$$

Question 3

a) Product rule

$$\frac{dy}{dx} = u\frac{dv}{du} + v\frac{du}{dv}$$

Let $u = 4\cos x - 3\sin x$ $\qquad$ $v = e^{3x-5}$

$\frac{du}{dx} = -4\sin x - 3\cos x$ $\qquad$ $\frac{dv}{dx} = 3e^{3x-5}$

$$\frac{d(uv)}{dx} = (4\cos x - 3\sin x)3e^{3x-5} + e^{3x-5}(-4\sin x - 3\cos x)$$

$$= e^{3x-5}(12\cos x - 9\sin x - 4\sin x - 3\cos x)$$

$$= \boxed{e^{3x-5}(9\cos x - 13\sin x)}$$

b)

Let $u = x^3 - 4x^2 + 7$ $\quad v = \ln x$ $\qquad \frac{dy}{dx} = u\frac{dv}{du} + v\frac{du}{dv}$

$\frac{du}{dx} = 3x^2 - 8x$ $\qquad \frac{dv}{dx} = \frac{1}{x}$

$$\frac{d(uv)}{dx} = \frac{(x^3 - 4x^2 + 7)}{x} + (\ln x)(3x^2 - 8x)$$

$$= \boxed{x^2 - 4x + \frac{7}{x} + (\ln x)(3x^2 - 8x)}$$

Question 4

e^u $\qquad u = -3x$ $\qquad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{d(e^u)}{du} = e^u$ $\qquad \frac{du}{dx} = -3$

$$\frac{d(e^{-3x})}{dx} = \frac{d(e^u)}{du} \times \frac{du}{dx}$$ chain rule

$$= e^u \times (-3)$$

$$= -3e^{-3x}$$

$$\frac{dy}{dx} = -3e^{-3x} + \frac{1}{x}$$

sub $x = 1$

$$\frac{dy}{dx} = -3e^{-3(1)} + \frac{1}{1} = 1 - 3e^{-3}$$

$$m_{normal} = -\frac{1}{\frac{dy}{dx}} = \frac{-1}{1 - 3e^{-3}} = \boxed{-1.176} \text{ (3dp)}$$

Question 5

Let $y = e^u$ $\quad u = 3x^2 + 5x - 2$ $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = e^u$ $\quad \frac{du}{dx} = 6x + 5$

$\frac{dy}{dx} = (6x+5)e^u$ $\quad$ Chain rule

$= (6x+5)e^{3x^2+5x-2}$

At $x = -2$

$\frac{dy}{dx} = (6(-2)+5)e^{3(-2)^2+5(-2)-2}$

$= -7e^0 = -7 = m$

$y - (1) = -7(x - (-2))$

$y - 1 = -7x - 14$

$\boxed{y + 7x + 13 = 0}$

Question 6

$$f(2) = \frac{g(2)}{h(2)}$$

$$f(2) = -4$$

point : $(2, -4)$

Quotient rule

$y = \frac{u}{v} \longrightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (in formula booklet)

$u = g(2) = 4$ $\quad$ $v = h(2) = -1$

$\frac{du}{dx} = g'(2) = 0$ $\quad$ $\frac{dv}{dx} = h'(2) = 2$

$$f'(2) = \frac{\cancel{(-1)(0)} - (4)(2)}{(-1)^2}$$

$$f'(2) = -8$$

Sub $(2, -4)$ and $m = -8$ into $y - y_1 = m(x - x_1)$.

$$y + 4 = -8(x - 2)$$

$$y = -8x + 16 - 4$$

$$\boxed{y = -8x + 12}$$

Question 7

a) Differenciate once

$$\frac{dy}{dx} = \boxed{3x^2 - 12}$$

... and differenciate again!

$$\frac{d^2y}{dx^2} = \boxed{6x}$$

b) Find x at the stationary points

$$\frac{dy}{dx} = 3x^2 - 12 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Two values since $\frac{dy}{dx} = 0$ at local min & local max.

Classify stationary points

At $x = 2$ $\quad \frac{d^2y}{dx^2} = 6(2) = 12 > 0 \quad \therefore$ local min

At $x = -2$ $\quad \frac{d^2y}{dx^2} = 6(-2) = -12 < 0 \quad \therefore$ local max

$$y = 2^3 - 12(2) + 7 = -9$$

LOCAL MINIMUM: $\boxed{(2, -9)}$

Question 8

a) Product rule $\quad \frac{dy}{dx} = u\frac{dv}{du} + v\frac{du}{dv}$

Let $u = x^2 - 1$ $\quad v = \ln(x+3)$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = \frac{1}{x+3}$

$$\boxed{f'(x) = \frac{(x^2-1)}{x+3} + (\ln(x+3))2x}$$

b) At A, the curve intersects the x axis, so

$$y = f(x) = 0$$

$$(x^2-1)\ln(x+3) = 0$$

$$(x+1)(x-1)\underbrace{\ln(x+3)}_{\substack{\ln 1 = 0 \\ x+3=1 \\ x=-2}} = 0$$

$x = -1, 1$ $\qquad x = -2$

A is the most negative point of intersection,

$$\therefore \boxed{A(-2, 0)}$$

c) $$y - y_1 = m(x - x_1)$$

When $x = -2$

$$m = f'(x) = \frac{(-2)^2 - 1}{(-2)+3} + 2(-2)\ln(-2+3)$$

$$= -4\ln 1 + 3 = 3$$

Sub in $x_1 = -2$, $y_1 = 0$ and $m = 3$

$$y - (0) = 3(x - (-2))$$

$$\boxed{y = 3x + 6}$$

Question 9

a) Product rule (in formula booklet)

$$y = uv \longrightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$u = x^2$ $v = e^x$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx} = e^x$

$$\boxed{f'(x) = x^2 e^x + 2x e^x}$$

b) Product rule (in formula booklet)

$$y = uv \longrightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$f'(x) = x^2 e^x + 2x e^x$$

$$f'(x) = e^x (x^2 + 2x)$$

$u = e^x$ $v = x^2 + 2x$

$\frac{du}{dx} = e^x$ $\frac{dv}{dx} = 2x + 2$

$$f''(x) = e^x (2x + 2) + e^x (x^2 + 2x)$$

$$\boxed{f''(x) = e^x (x^2 + 4x + 2)}$$

c) Points of inflection occur when $f''(x) = 0$.

$$0 = e^{x}(x^2 + 4x + 2)$$

$$0 = x^2 + 4x + 2$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ (in formula booklet)

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{8}}{2}$$

$$x = \frac{\overset{-2}{\cancel{-4}} \pm \cancel{2}\sqrt{2}}{\cancel{2}}$$

$$\boxed{x = -2 \pm \sqrt{2}}$$

d) $\lim_{x \to -2} x^2 e^{x} = f(-2) = (-2)^2 e^{(-2)}$

$$\lim_{x \to -2} x^2 e^{x} = f(-2) = \frac{4}{e^2}$$

$$\lim_{x \to -2} x^2 e^{x} = f(-2) = 0.54134\ldots$$

$$\boxed{\lim_{x \to -2} x^2 e^{x} = f(-2) \quad 0.541}$$

Question 10

a) Graph $f(x)$ on your GDC and count the number of points the gradient is 0 in the given domain.

$\boxed{\text{3 points}}$

b) Chain rule

$y = g(u)$, where $u = f(x)$

$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}$ (in formula booklet)

$u = 2\cos x$ $\qquad$ $y = 2e^{u}$

$\frac{du}{dx} = -2\sin x$ $\qquad$ $\frac{dy}{du} = 2e^{u}$

$f'(x) = 2e^{2\cos x} \times -2\sin x$

$f'(x) = -4\sin x \, e^{2\cos x}$

Sub $x = \pi/2$ into $f'(x)$.

$f'(\pi/2) = -4\sin(\pi/2)e^{2\cos(\pi/2)}$

$f'(\pi/2) = -4(1)e^{2(0)}$

$\boxed{f'(\pi/2) = -4}$

c) $f'(x) = -4\sin x \, e^{2\cos x}$

Sub $x = 3\pi/2$ into $f'(x)$

$f'(3\pi/2) = -4\sin(3\pi/2)e^{2\cos(3\pi/2)}$

$f'(3\pi/2) = -4(-1)e^{2(0)}$

$\boxed{f'(3\pi/2) = 4}$

d) $f(x)$ has 3 stationary points in the given domain $(-\pi, 0, \pi)$.

$f(2x)$ has 5 stationary points in the given domain $(-\pi, -\pi/2, 0, \pi/2, \pi)$

$\boxed{\therefore \text{rule} : 2k+1}$

Question 11

Derivative of $\sin x$ (in formula booklet)

$f(x) = \sin x \longrightarrow f'(x) = \cos x$

Chain rule

$g(x) = \sin^2 x$

$g(x) = (\sin x)^2 \longrightarrow g'(x) = 2\sin x\cos x$

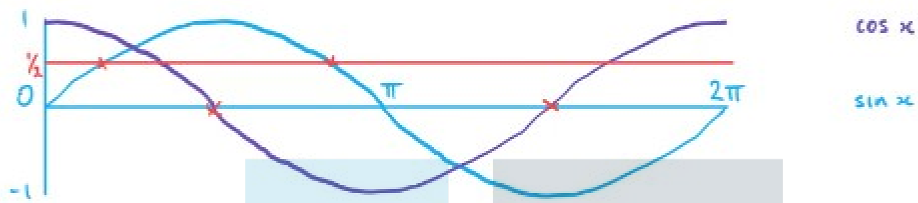
$f'(x) = g'(x)$

when $\cos x = 0$, $f'(x) = g'(x) = 0$.

$\cos x = 2\sin x \cos x$

$1 = 2\sin x$

$\sin x = 1/2$



$$\boxed{\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}}$$