



5.2 Integration

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5.2.1 Trapezoid Rule: Numerical Integration

Trapezoid Rule: Numerical Integration

What is the trapezoid rule?

- The trapezoidal rule is a numerical method used to find the approximate area enclosed by a curve, the X-axis and two vertical lines
 - it is also known as 'trapezoid rule' and 'trapezium rule'
- The trapezoidal rule finds an **approximation** of the area by **summing of the areas** of trapezoids beneath the curve

•
$$y_0 = f(a), \quad y_1 = f(a+h), \quad y_2 = f(a+2h) \text{ etc}$$

$$\int_a^b f(x) \, dx \approx \frac{1}{2} h \left[\left(y_0 + y_n \right) + 2 \left(y_1 + y_2 + \dots + y_{n-1} \right) \right]$$
where $h = \frac{b-a}{n}$

- Note that there are n trapezoids (also called strips) but (n+1) function values (y_{n})
- The trapezoidal rule is given in the formula booklet

What else can I be asked to do with the trapezoid rule?

- Comparing the true answer with the answer from the trapezoid rule
 - This may involve finding the **percentage error** in the approximation
 - The true answer may be given in the question, found from a GDC or from work on integration





b) Given that the area bounded by the curve, the *X*-axis and the lines x = 0 and x = 4 is 6.993 to three decimal places, calculate the percentage error in the trapezoidal rule approximation.







5.2.2 Introduction to Integration

Introduction to Integration

What is integration?

- Integration is the opposite to differentiation
 - Integration is referred to as antidifferentiation
 - The result of integration is referred to as the **antiderivative**
- Integration is the process of finding the expression of a function (antiderivative) from an expression of the derivative (gradient function)

What is the notation for integration?

An integral is normally written in the form

 $\int f(x) dx$

- the large operator means "integrate"
- "dx" indicates which variable to integrate with respect to
- f(x) is the function to be integrated (sometimes called the integrand)
- The antiderivative is sometimes denoted by F(x)
 - there's then no need to keep writing the whole integral; refer to it as F(x)
- F(x) may also be called the **indefinite integral** of f(x)

What is the constant of integration?

- Recall one of the special cases from Differentiating Powers of x
 - If f(x) = a then f'(x) = 0
- This means that integrating 0 will produce a **constant** term in the antiderivative
 - a zero term wouldn't be written as part of a function
 - every function, when integrated, potentially has a constant term
- This is called the ${f constant}$ of ${f integration}$ and is usually denoted by the letter ${f {\cal C}}$
 - it is often referred to as "plus *C*"
- Without more information it is impossible to deduce the value of this constant
 - there are endless antiderivatives, F(x), for a function f(x)



Integrating Powers of x

How do l integrate powers of x?

• Powers of X are integrated according to the following formula:

• If
$$f(x) = x^n$$
 then $\int f(x) dx = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Z}$, $n \neq -1$ and c is the constant of integration

- This is given in the formula booklet
- If the power of is X multiplied by a constant then the integral is also multiplied by that constant

• If
$$f(x) = ax^n$$
 then $\int f(x) dx = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Z}$, $n \neq -1$, a is a constant and c is

- the constant of integration

-

- notation can still be used with integration
- Note that the formulae above do not apply when n = -1 as this would lead to division by zero
- Don't forget the special case:

$$\int a \, \mathrm{d}x = ax + c$$

e.g.
$$\int 4 \, dx = 4x + c$$

- This allows constant terms to be integrated
- Functions involving fractions with denominators in terms of X will need to be rewritten as negative powers of X first
 - e.g. If $f(x) = \frac{4}{x^2}$ then rewrite as $f(x) = 4x^{-2}$ and integrate

How do lintegrate sums and differences of powers of x?

2

The formulae for integrating powers of X apply to all integers so it is possible to integrate any expression that is a sum or difference of powers of X

e.g. If
$$f(x) = 8x^3 - 2x + 4$$
 then

$$\int f(x) dx = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$$

Products and quotients cannot be integrated in this way so would need expanding/simplifying first

e.g. If
$$f(x) = 8x^2(2x-3)$$
 then

$$\int f(x) \, dx = \int (16x^3 - 24x^2) \, dx = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$$





Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^4 - 2x^2 + 3 - \frac{1}{x^4}$$

find an expression for Y in terms of X.





5.2.3 Applications of Integration

Finding the Constant of Integration

What is the constant of integration?

- When finding an **anti-derivative** there is a constant term to consider
 - this constant term, usually called ${\cal C}$, is the **constant** of **integration**
- In terms of graphing an anti-derivative, there are endless possibilities
 - collectively these may be referred to as the family of antiderivatives or family of curves
 - the constant of integration is determined by the **exact** location of the curve
 - if a point on the curve is known, the constant of integration can be found

How do I find the constant of integration?

• For $F(x) + c = \int f(x) dx$, the **constant** of **integration**, c - and so the particular **antiderivative** - can be found if a point the graph of y = F(x) + c passes through is known

STEP 1

If need be, rewrite f(x) into an integrable form Each term needs to be a power of X (or a constant)

STEP 2

Integrate each term of f'(x), remembering the constant of integration, "+c" (Increase power by 1 and divide by new power)

STEP 3

Substitute the *X* and *Y* coordinates of a given point in to F(x) + c to form an equation in *c* Solve the equation to find *c*



Worked example

The graph of y = f(x) passes through the point (3, -4). The gradient function of f(x) is given by $f'(x) = 3x^2 - 4x - 4$.

Find f(x).

STEP 1
$$f'(x)$$
 is already in an integrable form
 $f'(x) = 3x^2 - 4x - 4$
STEP 2 Integrate, remembering "+c"
 $f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} - 4x + c$
 $f(x) = x^3 - 2x^2 - 4x + c$
STEP 3 Substitute x and y coordinates to find c
 $f(3) = -4$
 $\therefore (3)^3 - 2(3)^2 - 4(3) + c = -4$
 $27 - 18 - 12 + c = -4$
 $c = -1$
 $\therefore f(x) = x^3 - 2x^2 - 4x - 1$



Area Under a Curve Basics

What is meant by the area under a curve?

- The phrase "area under a curve" refers to the area bounded by
 - the graph of y = f(x)
 - the X-axis
 - the vertical line x = a
 - the vertical line x = b
- The exact area under a curve is found by evaluating a definite integral
- The graph of y = f(x) could be a straight line
 - the use of integration described below would still apply
 - but the shape created would be a trapezoid

• so it is easier to use "
$$A = \frac{1}{2}h(a+b)$$

What is a definite integral?

$$\int_{a}^{b} f(x) \, \mathrm{d}x = F(b) - F(a)$$

- This is known as the Fundamental Theorem of Calculus
- **a** and **b** are called limits
 - **a** is the **lower** limit
 - **b** is the **upper** limit
- f(x) is the integrand
- F(x) is an **antiderivative** of f(x)
- The constant of integration ("+ C") is not needed in definite integration
 - "+C" would appear alongside both F(a) and F(b)
 - subtracting means the "+ C"'s cancel



How do I form a definite integral to find the area under a curve?

• The graph of y = f(x) and the *x*-axis should be obvious boundaries for the area so the key here is in finding *a* and *b* - the **lower** and **upper** limits of the **integral**

STEP 1

Use the given sketch to help locate the limits You may prefer to plot the graph on your GDC and find the limits from there

STEP 2

Look carefully where the 'left' and 'right' boundaries of the area lie If the boundaries are vertical lines, the limits will come directly from their equations Look out for the *Y*-axis being one of the (vertical) boundaries – in this case the limit (*X*) will be 0 One, or both, of the limits, could be a root of the equation f(x) = 0i.e. where the graph of y = f(x) crosses the *X*-axis In this case solve the equation f(x) = 0 to find the limit(s)

A GDC will solve this equation, either from the graphing screen or the equation solver

STEP 3

The definite integral for finding the area can now be set up in the form

$$A = \int_{a}^{b} f(x) \, \mathrm{d}x$$



Definite Integrals using GDC

Does my calculator/GDC do definite integrals?

 Modern graphic calculators (and some 'advanced' scientific calculators) have the functionality to evaluate definite integrals

- i.e. they can calculate the **area under a curve** (see above)
- If a calculator has a button for evalutaing definite integrals it will look something like
- This may be a physical button or accessed via an on-screen menu
- Some GDCs may have the ability to find the area under a curve from the graphing screen
- Be careful with any calculator/GDC, they may not produce an exact answer

How do I use my GDC to find definite integrals?

Without graphing first ...

- Once you know the definite integral function your calculator will need three things in order to evaluate it
 - The function to be integrated (integrand) (f(x))
 - The **lower** limit (a from x = a)
 - The **upper** limit (b from x = b)
- Have a play with the order in which your calculator expects these to be entered some do not always work left to right as it appears on screen!

With graphing first ...

- Plot the graph of V = f(X)
 - You may also wish to plot the vertical lines x = a and x = b
 - make sure your GDC is expecting an "X = " style equation"
 - Once you have plotted the graph you need to look for an option regarding "area" or a physical button
 - it may appear as the integral symbol (e.g. $\int dx$)
- - your GDC may allow you to select the lower and upper limits by moving a cursor along the curve - however this may not be very accurate
 - your GDC may allow you to type the exact limits required from the keypad
 - the lower limit would be typed in first
 - read any information that appears on screen carefully to make sure



Worked example

a) Using your GDC to help, or otherwise, sketch the graphs of $y = x^4 - 2x^2 + 5$, x = 1 and x = 2

x = 2 on the same diagram

Use the 'graph' menu on your GOC to plot $y = x^4 - 2x^2 + 5$. You may then need to change the 'input type' to 'x='

to enter x=1 and x=2.

Plot the graph on your GDC and sketch the result, ensuring to include all the main properties of each graph.



b)

The area enclosed by the three graphs from part (a) and the X-axis is to be found. Write down an integral that would find this area.



c) Using your GDC, or otherwise, find the exact area described in part (b).

Give your answer in the form $\frac{a}{b}$ where a and b are integers.



Area = $\int_{1}^{2} (x^{4} - 2x^{2} + 5) dx = \frac{98}{15}$ square units

From the graphing screen on our GOC the integral value was given as 6.53333333 - not exact!

