DP IB Maths: AA HL

5.2 Further Differentiation

Contents

5.2.1 Differentiating Special Functions

Differentiating Trig Functions

How do I differentiate sin, cos and tan?

- The derivative of $y = \sin\ x$ is $\frac{dy}{dx} = \cos\ x$
- The derivative of $y = \cos\ x$ is $\frac{dy}{dx} = -\sin\ x$
- The derivative of $y = \tan\ x$ is $\frac{dy}{dx} = \sec^2 x$
  - This result can be derived using **quotient rule**
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = \sin(ax+b)$ is $\frac{dy}{dx} = a\cos(ax+b)$
  - the derivative of $y = \cos(ax+b)$ is $\frac{dy}{dx} = -a\sin(ax+b)$
  - **the derivative of** $y = \tan(ax+b)$ **is** $\frac{dy}{dx} = a\sec^2(ax+b)$
- For the **general** function $f(x)$,
  - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
  - the derivative of $y = \tan(f(x))$ is $\frac{dy}{dx} = f'(x)\sec^2(f(x))$
- These last three results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
  - Ensure you know how to change the angle mode on your GDC

Worked example

a) Find $f'(x)$ for the functions

i. $f(x) = \sin x$
ii. $f(x) = \cos(5x + 1)$

i. $\boxed{f'(x) = \cos x}$

ii. $\boxed{f'(x) = -5\sin(5x+1)}$ (Linear function $ax+b$)

b) A curve has equation $y = \tan\left(6x^2 - \frac{\pi}{4}\right)$.

Find the gradient of the tangent to the curve at the point where $x = \frac{\sqrt{\pi}}{2}$.

Give your answer as an exact value.

This is of the form $y = \tan(f(x))$

so $\frac{dy}{dx} = f'(x)\sec^2(f(x))$

$f(x) = 6x^2 - \frac{\pi}{4}$

$\therefore f'(x) = 12x$

$\therefore \frac{dy}{dx} = 12x\sec^2\left(6x^2 - \frac{\pi}{4}\right)$

At $x = \frac{\sqrt{\pi}}{2}$, $\frac{dy}{dx} = 12\left(\frac{\sqrt{\pi}}{2}\right)\sec^2\left[6\left(\frac{\sqrt{\pi}}{2}\right)^2 - \frac{\pi}{4}\right]$

$= \frac{6\sqrt{\pi}}{\cos^2\left(\frac{5\pi}{4}\right)}$ $\left(\sec^2 x = \frac{1}{\cos^2 x}\right)$

$\boxed{\therefore \frac{dy}{dx} = 12\sqrt{\pi} \text{ at } x = \frac{\sqrt{\pi}}{2}}$

Differentiating e^x & lnx

How do I differentiate exponentials and logarithms?

- The derivative of $y = \mathrm{e}^x$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ where $x > 0$
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = \mathrm{e}^{(ax+b)}$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = a\mathrm{e}^{(ax+b)}$
  - the derivative of $y = \ln(ax+b)$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{(ax+b)}$
    - in the special case $b = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ (a's cancel)
- For the **general** function $\mathrm{f}(x)$,
  - the derivative of $y = \mathrm{e}^{\mathrm{f}(x)}$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$
  - the derivative of $y = \ln(\mathrm{f}(x))$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$
- The last two sets of results can be derived using the **chain rule**

Worked example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where $x = 2$ giving your answer in the form $y = a + be^{c}$, where $a,\ b$ and c are integers to be found.

$y = e^{-3x+1} + 2(\ln 5x)$

$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"

"$y = \ln(ax+b)$, special case $b=0$, $\frac{dy}{dx} = \frac{1}{x}$"

At $x = 2$, $\frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$

Your GDC may be able to find gradients but probably not in the form required. It is still helpful to check approximate answers though.

$\therefore$ Gradient at $x = 2$ is $1 - 3e^{-5}$

i.e. $a = 1$, $b = -3$, $c = -5$

5.2.2 Techniques of Differentiation

Chain Rule

What is the chain rule?

- The **chain rule** states if y is a function of u and u is a function of x then

$$y = f(u(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

  - This is given in the **formula booklet**

- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x))g'(x)$$

How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually x) does not 'appear alone'
    - $\sin x$ – **not** a composite function, x 'appears alone'
    - $\sin(3x+2)$ **is a composite function;** x is tripled and has 2 added to it before the sine function is applied

How do I use the chain rule?

STEP 1
Identify the two functions
Rewrite y as a function of u; $y = f(u)$
Write u as a function of x; $u = g(x)$

STEP 2

Differentiate y with respect to u to get $\frac{\mathrm{d}y}{\mathrm{d}u}$

Differentiate u with respect to x to get $\frac{\mathrm{d}u}{\mathrm{d}x}$

STEP 3

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$ and substitute u back in for $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

Are there any standard results for using chain rule?

- There are **five** general results that can be useful
  - If $y = (f(x))^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = n\mathrm{f}'(x)\mathrm{f}(x)^{n-1}$
  - If $y = \mathrm{e}^{f(x)}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)e^{f(x)}$
  - If $y = \ln(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)}{f(x)}$
  - If $y = \sin(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)\cos(f(x))$
  - If $y = \cos(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -f'(x)\sin(f(x))$

Worked example

a) Find the derivative of $y = (x^2 - 5x + 7)^7$.

STEP 1 Identify the two functions and rewrite

$y = u^7$ i.e. $f(u) = u^7$

$u = x^2 - 5x + 7$ i.e. $g(x) = x^2 - 5x + 7$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

$\frac{dy}{du} = 7u^6$ $\frac{du}{dx} = 2x - 5$

STEP 3 Apply chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$\frac{dy}{dx} = 7u^6(2x-5)$

and substitute u back for g(x)

$$\boxed{\frac{dy}{dx} = 7(2x-5)(x^2-5x+7)^6}$$

b) Find the derivative of $y = \sin(e^{2x})$.

EXAM PAPERS PRACTICE

$y = \sin(e^{2x})$ "... differentiate $\sin \square$, ignore e^{2x}..."

$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$ "... multiply by derivative of e^{2x} ..."

" $y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$ "
or by applying chain rule again

$$\therefore \boxed{\frac{dy}{dx} = 2e^{2x}\cos(e^{2x})}$$

Product Rule

What is the product rule?

- The **product rule** states if y is the product of two functions $u(x)$ and $v(x)$ then

$$y = uv$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g'(x) + g(x)f'(x)$$

- **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of x"
    - $\sin x \cos x$ is a product, "sin *x* times cos x"

How do I use the product rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the product rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v'

Worked example

a) Find the derivative of $y = \mathrm{e}^x \sin x$.

$y = e^x \sin x$

STEP 1 Identify functions and differentiate

$u = e^x \quad v = \sin x$

$u' = e^x \quad v' = \cos x$

arranging u, v, u', v' in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule: '$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$'

(As it is given in the formula booklet)

$y' = e^x \cos x + e^x \sin x$

$$\boxed{\therefore \frac{dy}{dx} = e^x(\cos x + \sin x)}$$

It is straightforward to take a factor of e^x out

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$y = 5x^2 \cos 3x^2$

STEP 1 $u = 5x^2 \quad v = \cos 3x^2$

$u' = 10x \quad v' = -\sin 3x^2 \times 6x$ chain rule

$v' = -6x \sin 3x^2$

STEP 2 $y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$$\boxed{\therefore \frac{dy}{dx} = 10x(\cos 3x^2 - 3x^2 \sin 3x^2)}$$

Quotient Rule

What is the quotient rule?

- The **quotient rule** states if y is the quotient $\frac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of x
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

How do I use the quotient rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the quotient rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important

Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v',

Worked example

Differentiate $f(x) = \dfrac{\cos 2x}{3x+2}$ with respect to x.

STEP 1 Identify u and v, differentiate

$u = \cos 2x \qquad v = 3x+2$

$u' = -2\sin 2x \qquad v' = 3$

chain rule

opposite diagonals match up

STEP 2 Apply quotient rule: ' $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ '

(As it is given in the formula booklet)

$$f'(x) = \frac{(3x+2)(-2\sin 2x) - (\cos 2x)(3)}{(3x+2)^2}$$

$$\therefore f'(x) = \boxed{\frac{-2(3x+2)\sin 2x - 3\cos 2x}{(3x+2)^2}}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)

5.2.3 Higher Order Derivatives

Second Order Derivatives

What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = \mathrm{f}(x)$
  - $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$ (First order derivative)
  - $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{f}''(x)$ (Second order derivative)
- Note the position of the superscript 2's
  - **d**ifferentiating twice (so $\mathbf{d}^2$) with respect to x twice (so x^2)
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate** of **change** of a function
  - a **second order derivative** is the **rate** of **change** of the **rate** of **change** of a function
    - i.e. the **rate** of **change** of the function's **gradient**
- **Second order derivatives** can be used to
  - test for local minimum and maximum points
  - help determine the nature of stationary points
  - help determine the concavity of a function
  - graph derivatives

How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions, roots,** etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule

Worked example

Given that $f(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a) Find $f'(x)$ and $f''(x)$.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ REWRITE AS POWERS OF x

$f'(x) = -(\frac{1}{2})x^{\frac{1}{2}-1} + 3(-\frac{1}{2})x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$ DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}(-\frac{1}{2})x^{-\frac{1}{2}-1} - \frac{3}{2}(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$ DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b) Evaluate $f''(3)$.

Give your answer in the form $a\sqrt{b}$, where b is an integer and a is a rational number.

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$ $x^{\frac{3}{2}} = x\sqrt{x}$ $x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$

RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$

Higher Order Derivatives

What is meant by higher order derivatives of a function?

- Many functions can be **differentiated** numerous times
  - The third, fourth, fifth, etc derivatives of a function are generally called **higher order derivatives**
- It may not be possible, or practical to (algebraically) differentiate complicated functions more than once or twice
- **Polynomials** will, eventually, have higher order derivatives of zero
  - Since powers of x reduce by 1 each time

What is the notation for higher order derivatives?

- The notation for higher order derivatives follows the logic from the first and second derivatives

$$f^{(n)}(x) \quad \text{or} \quad \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

  except the 'dash' (prime) notation is replaced with numbers as this would become cumbersome after the first few
  - e.g. the fifth derivative would be

$$f^{(5)}(x) \quad \text{or} \quad \frac{\mathrm{d}^5 y}{\mathrm{d}x^5}$$

How do I find a higher order derivative of a function?

- By differentiating as many times as required!
- This may involve
  - rewriting fractions, roots, etc as negative and/or fractional powers
  - differentiating trigonometric functions, exponentials and logarithms
  - using chain rule
  - using product or quotient rule

Worked example

It is given that $f(x) = \sin 2x$.

a) Show that $f^4(x) = 16f(x)$.

$f^4(x)$ is the FOURTH derivative

$f(x) = \sin 2x$

$f'(x) = 2\cos 2x$ ('sin → cos', chain rule)

$f''(x) = -4\sin 2x$ ('cos → -sin', chain rule)

$f^3(x) = -8\cos 2x$ You should notice a pattern by now...

$f^4(x) = 16\sin 2x$

$$\boxed{\therefore\ f^4(x) = 16\sin 2x = 16f(x) \text{ as required}}$$

b) Without further working, write down an expression for $f^8(x)$.

We can see from part (a)

- the coefficient of each derivative is a power of 2
- $\sin 2x$ ($f(x)$) is involved in every even derivative
- $\sin 2x$ is positive in every other even derivative

$$\boxed{\therefore\ f^8(x) = 256\sin 2x}$$

5.2.4 Further Applications of Differentiation

Stationary Points & Turning Points

What is the difference between a stationary point and a turning point?

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** is a type of stationary point, but in addition the **function changes** from **increasing** to **decreasing,** or **vice versa**
  - The curve **'turns'** from **'going upwards'** to **'going downwards'** or **vice versa**
  - **Turning points** will either be **(local) minimum** or **maximum** points
- A **point** of **inflection** ***could*** also be a **stationary point** but is **not** a turning point

How do I find stationary points and turning points?

- For the function $y = f(x)$, **stationary points** can be found using the following process

  STEP 1

  Find the **gradient function,** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$

  STEP 2

  Solve the equation $f'(x) = 0$ to find the x-coordiante(s) of any stationary points

  STEP 3

  If the y-coordinates of the stationary points are also required then substitute the x-coordinate(s) into $f(x)$

- A GDC will solve $f'(x) = 0$ and most will find the coordinates of turning points (minimum and maximum points) in graphing mode

Testing for Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
  - i.e. $f'(x) = 0$
- A **local minimum** point, $(x, f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x, f(x))$ will be the highest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend** to **infinity** for **large** values of x (and/or **minus infinity** for **large negative** values of x)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of $f(x)$ for **all values** of x
  - similar for a **global** maximum point

How do I find local minimum & maximum points?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function $f(x)$ …

  STEP 1

  Find $f'(x)$ and solve $f'(x) = 0$ to find the x-coordinates of any stationary points

  STEP 2 (Second derivative)

  Find $f''(x)$ and evaluate it at each of the stationary points found in **STEP 1**

  STEP 3 (Second derivative)

  - If $f''(x) = 0$ then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
  - If $f''(x) > 0$ then the curve of the graph of $y = f(x)$ is **concave up** and the stationary point is a **local minimum** point
  - If $f''(x) < 0$ then the curve of the graph of $y = f(x)$ is **concave down** and the stationary point is a **local maximum** point

  STEP 4 (First derivative)

  Find the sign of the first derivative just either side of the stationary point; i.e. evaluate $f'(x-h)$ and $f'(x+h)$ for small h

- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h)<0,\ f'(x)=0,\ f'(x+h)>0$
- A **local maximum point** changes the **function** from **increasing** to **decreasing**
  - the **gradient** changes from **positive** to **negative**
  - $f'(x-h)>0,\ f'(x)=0,\ f'(x+h)<0$

- A **stationary point** of **inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not change** sign
  - $f'(x-h)>0,\ f'(x+h)>0$ or $f'(x-h)<0,\ f'(x+h)<0$
  - a **point** of **inflection** does **not** necessarily have $f'(x)=0$
    - this method will only find those that do - and are often called **horizontal points** of **inflection**

Worked example

Find the coordinates and the nature of any stationary points on the graph of $y=f(x)$ where $f(x)=2x^3-3x^2-36x+25$.

At stationary points, $f'(x)=0$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$6(x^2 - x - 6) = 0$

$(x-3)(x+2) = 0$

$x = 3$, $\quad y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$

$x = -2$, $\quad y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$

Using the second derivative to determine their nature

$f''(x) = 12x - 6 = 6(2x-1)$

$f''(3) = 6(2\times 3 - 1) = 30 > 0$

$\therefore x = 3$ is a local minimum point

$f''(-2) = 6(2\times -2 - 1) = -30 < 0$

$\therefore x = -2$ is a local maximum point

(Note: In this case, both stationary points are turning points)

Turning points are:

$(3, -56)$ local minimum point

$(-2, 69)$ local maximum point

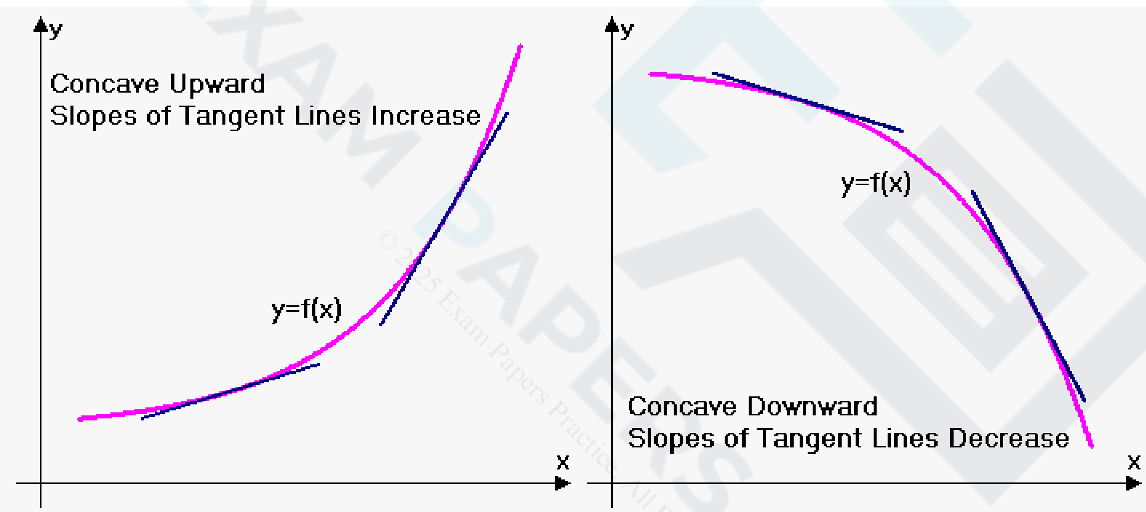
Use a GDC to graph $y = f(x)$ and the max/min solving feature to check the answers.

5.2.5 Concavity & Points of Inflection

Concavity of a Function

What is concavity?

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if $f''(x) < 0$ for all values of x in an interval
  - a curve is **CONCAVE UP** if $f''(x) > 0$ for all values of x in an interval



Worked example

The function $f(x)$ is given by $f(x) = x^3 - 3x + 2$.

a) Determine whether the curve of the graph of $y = f(x)$ is concave down or concave up at the points where $x = -2$ and $x = 2$.

$f(x) = x^3 - 3x + 2$
$f'(x) = 3x^2 - 3$
$f''(x) = 6x$

$f''(-2) = 6 \times -2 = -12 < 0$ (concave down)

$f''(2) = 6 \times 2 = 12 > 0$ (concave up)

$\boxed{\text{At } x = -2,\ y = f(x) \text{ is concave down} \\ \text{At } x = 2,\ y = f(x) \text{ is concave up}}$

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answers are sensible

b) Find the values of x for which the curve of the graph $y = f(x)$ of is concave up.

$f''(x) = 6x$ from part (a)

Concave up is $f''(x) > 0$

$6x > 0$ when $x > 0$

$\boxed{\therefore\ y = f(x) \text{ is concave up for } x > 0}$

Use your GDC to check your answer

Points of Inflection

What is a point of inflection?

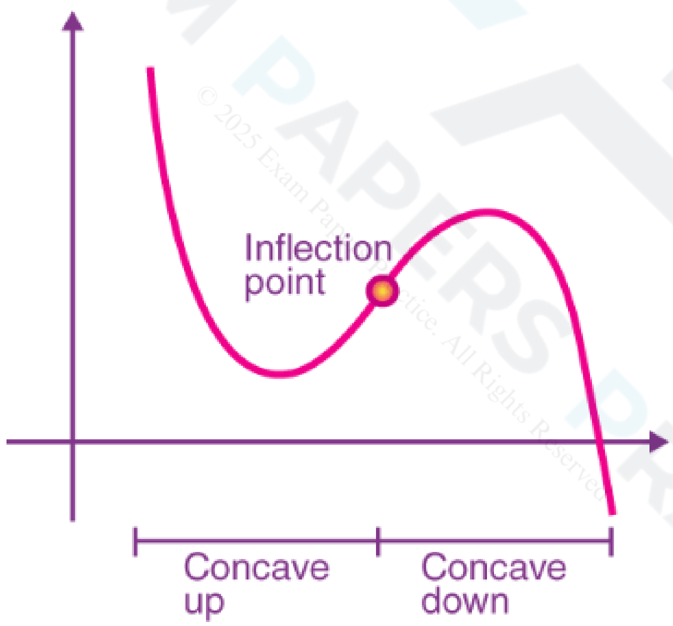
- A point at which the curve of the graph of $y = f(x)$ changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion,** may sometimes be used

What are the conditions for a point of inflection?

- A point of inflection requires **BOTH** of the following two conditions to hold
  - the **second derivative** is zero
    - $f''(x) = 0$

  AND

  - the graph of $y = f(x)$ changes **concavity**
    - $f''(x)$ changes **sign** through a **point** of **inflection**



- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
  - points where $f''(x) = 0$ could be **local minimum** or **maximum** points
    - the **first derivative** test would be needed
  - However, if it is already known $f(x)$ has a point of inflection at $x = a$, say, then $f''(a) = 0$

What about the first derivative, like with turning points?

- A **point** of **inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ($f'(x) = 0$)
  - If it does, it is also a **stationary point** and is often called a **horizontal point** of **inflection**
    - the tangent to the curve at this point would be horizontal
  - The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

How do I find the coordinates of a point of inflection?

- For the function $f(x)$

  STEP 1

  Differentiate $f(x)$ **twice** to find $f''(x)$ and solve $f''(x) = 0$ to find the x-coordinates of possible points of inflection

  STEP 2

  Use the **second derivative** to **test** the **concavity** of $f(x)$ either side of $x = a$

  - If $f''(x) < 0$ then $f(x)$ is concave down
  - If $f''(x) > 0$ then $f(x)$ is concave up

  If concavity changes, $x = a$ is a **point of inflection**

  STEP 3

  If required, the y-coordinate of a point of inflection can be found by substituting the x-coordinate into $f(x)$

Worked example

Find the coordinates of the point of inflection on the graph of $y = 2x^3 - 18x^2 + 24x + 5$.
Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve $f''(x) = 0$

$f(x) = 2x^3 - 18x^2 + 24x + 5$

$f'(x) = 6x^2 - 36x + 24$

$f''(x) = 12x - 36$

$12x - 36 = 0$ when $x = 3$

STEP 2: Use the second derivative to test concavity

$f''(3) = 0$

$f''(2.9) < 0$ (concave down)

$f''(3.1) > 0$ (concave up)

$\therefore$ concavity changes through $x = 3$

STEP 3: The y-coordinate is required

$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

Since $f''(3) = 0$ **AND** the graph of $y = f(x)$ changes concavity through $x = 3$, the point $(3, -31)$ is a point of inflection.

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answer is sensible

5.2.6 Derivatives & Graphs

Derivatives & Graphs

How are derivatives and graphs connected?

- If the graph of a function $y = f(x)$ is known, or can be sketched, then it is also possible to sketch the graphs of the derivatives $y = f'(x)$ and $y = f''(x)$
- The key properties of a graph include
  - the **y-axis intercept**
  - the **x-axis intercepts** – the **roots** of the function; where $f(x) = 0$
  - **stationary points**; where $f'(x) = 0$
    - turning points – (local) **minimum** and **maximum** points
    - (horizontal) **points of inflection**
  - (non-stationary, $f'(x) \neq 0$) points of inflection
  - **asymptotes** – **vertical** and **horizontal**
  - intervals where the graph is **increasing** and **decreasing**
  - intervals where the graph is **concave down** and **concave up**
- **Not** all graphs have all of these properties and **not** all can be determined without knowing the expression of the function
- However questions will provide enough information to sketch
  - the **shape** of the graph
  - **some** of the **key properties** such as **roots** or **turning points**

How do I sketch the graph of *y* = f'(*x*) from the graph of *y* = f(*x*)?

- The graph of $y = f'(x)$ will have its
  - x-axis **intercepts** at the x-coordinates of the **stationary points** of $y = f(x)$
  - **turning points** at the x-coordinates of the **points of inflection** of $y = f(x)$
- For intervals where $y = f(x)$ is concave up, $y = f'(x)$ will be increasing
- For intervals where $y = f(x)$ is concave down, $y = f'(x)$ will be decreasing
- For intervals where $y = f(x)$ is increasing, $y = f'(x)$ will be positive
- For intervals where $y = f(x)$ is decreasing, $y = f'(x)$ will be negative

How do I sketch the graph of *y* = f''(*x*) from the graph of *y* = f(*x*)?

- First sketch the graph of $y = f'(x)$ from $y = f(x)$, as per the above process
- Then, using the same process, sketch the graph of $y = f''(x)$ from the graph of $y = f'(x)$
- There are a couple of things you can deduce about the graph of $y = f''(x)$ directly from the graph of $y = f(x)$

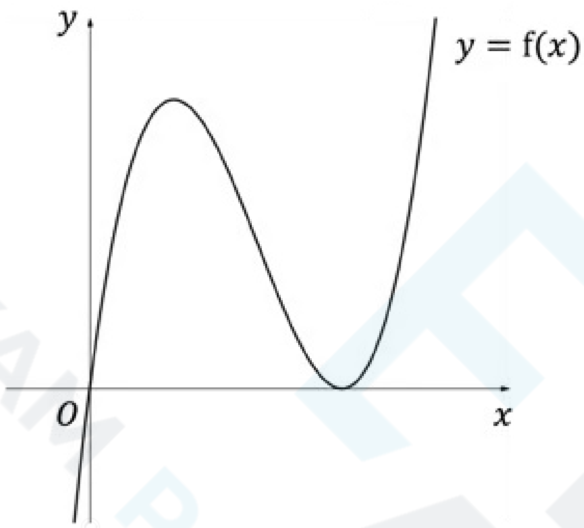
- The graph of $y = f''(x)$ will have its x-axis **intercepts** at the x-coordinates of the **points of inflection** of $y = f(x)$
- For intervals where $y = f(x)$ is concave up, $y = f''(x)$ will be positive
- For intervals where $y = f(x)$ is concave down, $y = f''(x)$ will be negative

Is it possible to sketch the graph **of y = f(x) from the graph of a derivative?**

- It is possible to **sketch** a graph of $y = f(x)$ by considering the reverse of the above
  - For intervals where $y = f'(x)$ is **positive**, $y = f(x)$ will be **increasing** but is **not** necessarily positive
  - For intervals where $y = f'(x)$ is **negative**, $y = f(x)$ will be **decreasing** but is **not** necessarily negative
  - **Roots** of $y = f'(x)$ give the x-coordinates of the **stationary points** of $y = f(x)$
- There are some properties of the graph of $y = f(x)$ that **cannot** be determined from the graph of $y = f'(x)$
  - the $\boldsymbol{y}$**-axis intercept**
  - the **intervals** for which $y = f(x)$ is **positive** and **negative**
  - the roots of $y = f(x)$
- Unless a **specific** point the **curve passes through** is **known**, the **constant** of **integration cannot** be determined
  - the *exact* location of the curve will remain unknown
  - but it will still be possible to **sketch** its **shape**
- If starting from the graph of the **second derivative**, $y = f''(x)$, it is easier to sketch the graph of $y = f'(x)$ first, then sketch $y = f(x)$

Worked example

The graph of $y = f(x)$ is shown in the diagram below.

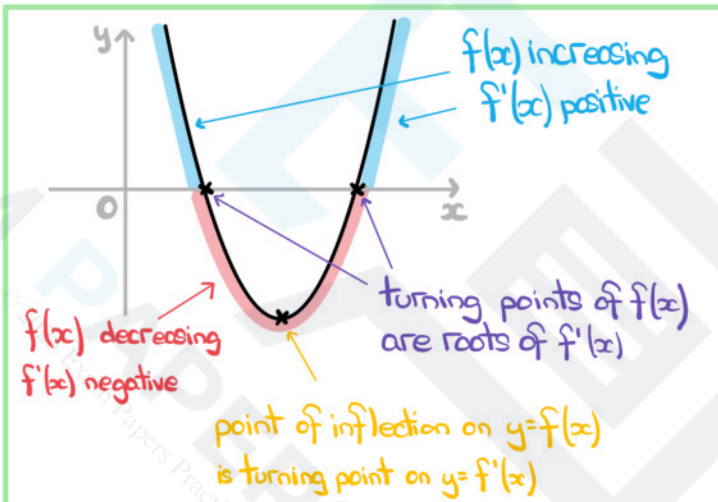


On separate diagrams sketch the graphs of $y = f'(x)$ and $y = f''(x)$, labelling any roots and turning points.

Key features from graph of $y=f(x)$ are:

- local maximum and local minimum
- $f(x)$ is increasing 'at either end'
- $f(x)$ is decreasing between max. and min.
- point of inflexion (non-stationary), graph changes from concave down to concave up

Graph of $y=f'(x)$:



Graph of $y=f''(x)$: