# DP IB Maths: AI HL

## 5.2 Further Differentiation

### Contents

## 5.2.1 Differentiating Special Functions

### Differentiating Trig Functions

**How do I differentiate sin, cos and tan?**

- The derivative of $y = \sin\ x$ is $\frac{dy}{dx} = \cos\ x$
- The derivative of $y = \cos\ x$ is $\frac{dy}{dx} = -\sin\ x$
- The derivative of $y = \tan\ x$ is $\frac{dy}{dx} = \frac{1}{\cos^2\ x}$
  - This result can be derived using **quotient rule**
- All three of these derivatives are given in the **formula booklet**
- For the **linear** function $ax + b$, where $a$ and $b$ are constants,
  - the derivative of $y = \sin(ax + b)$ is $\frac{dy}{dx} = a\cos(ax + b)$
  - the derivative of $y = \cos(ax + b)$ is $\frac{dy}{dx} = -a\sin(ax + b)$
  - **the derivative of** $y = \tan(ax + b)$ is $\frac{dy}{dx} = \frac{a}{\cos^2(ax + b)}$
- For the **general** function $f(x)$,
  - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
  - the derivative of $y = \tan(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{\cos^2(f(x))}$
- These last three results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
  - Ensure you know how to change the angle mode on your GDC

## Worked example

a) Find $f'(x)$ for the functions

i. $f(x) = \sin x$

ii. $f(x) = \cos(5x + 1)$

i. $f'(x) = \cos x$

ii. $f'(x) = -5\sin(5x+1)$ (Linear function $ax+b$)

b) A curve has equation $y = \tan\left(6x^2 - \frac{\pi}{4}\right)$.

Find the gradient of the tangent to the curve at the point where $x = \frac{\sqrt{\pi}}{2}$.

Give your answer as an exact value.

This is of the form $y = \tan(f(x))$

so $\frac{dy}{dx} = \frac{f'(x)}{\cos^2(f(x))}$

$f(x) = 6x^2 - \frac{\pi}{4}$

$\therefore f'(x) = 12x$

$\therefore \frac{dy}{dx} = \frac{12x}{\cos^2\left(6x^2 - \frac{\pi}{4}\right)}$

At $x = \frac{\sqrt{\pi}}{2}$, $\frac{dy}{dx} = \frac{12\left(\frac{\sqrt{\pi}}{2}\right)}{\cos^2\left[6\left(\frac{\sqrt{\pi}}{2}\right)^2 - \frac{\pi}{4}\right]}$

$= \frac{6\sqrt{\pi}}{\cos^2\left(\frac{5\pi}{4}\right)}$

$\therefore \frac{dy}{dx} = 12\sqrt{\pi}$ at $x = \frac{\sqrt{\pi}}{2}$

## Differentiating e^x & lnx

**How do I differentiate exponentials and logarithms?**

- The derivative of $y = e^x$ is $\frac{dy}{dx} = e^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$
- For the **linear** function $ax + b$, where $a$ and $b$ are constants,
    - the derivative of $y = e^{(ax+b)}$ is $\frac{dy}{dx} = ae^{(ax+b)}$
    - the derivative of $y = \ln(ax + b)$ is $\frac{dy}{dx} = \frac{a}{(ax+b)}$
        - in the special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$ ($a$'s cancel)
- For the **general** function $f(x)$,
    - the derivative of $y = e^{f(x)}$ is $\frac{dy}{dx} = f'(x)e^{f(x)}$
    - the derivative of $y = \ln(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
- The last two sets of results can be derived using the **chain rule**

### Worked example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where $x = 2$ gving your answer in the form $y = a + be^{c}$, where $a,\ b$ and $c$ are integers to be found.

$$y = e^{-3x+1} + 2(\ln 5x)$$

$$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$$

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"

"$y = \ln(ax+b)$, special case $b=0$, $\frac{dy}{dx} = \frac{1}{x}$"

At $x = 2$, $\frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$

Your GDC may be able to find gradients but probably not in the exact form required. It is still helpful to check approximate answers though.

$\therefore$ Gradient at $x = 2$ is $1 - 3e^{-5}$

i.e. $a = 1$, $b = -3$, $c = -5$

EXAM PAPERS PRACTICE

## 5.2.2 Techniques of Differentiation

## Chain Rule

### What is the chain rule?

- The **chain rule** states if $y$ is a function of $u$ and $u$ is a function of $x$ then

$$y = f(u(x))$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

### How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually $x$) does not 'appear alone'
    - $\sin x$ – **not** a composite function, $x$ 'appears alone'
    - $\sin(3x+2)$ **is a composite function;** $x$ is tripled and has 2 added to it before the sine function is applied

### How do I use the chain rule?

**STEP 1**
Identify the two functions
Rewrite $y$ as a function of $u$; $y = f(u)$
Write $u$ as a function of $x$; $u = g(x)$

**STEP 2**

Differentiate $y$ with respect to $u$ to get $\frac{dy}{du}$

Differentiate $u$ with respect to $x$ to get $\frac{du}{dx}$

**STEP 3**

Obtain $\frac{dy}{dx}$ by applying the formula $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and substitute $u$ back in for $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

## Are there any standard results for using chain rule?

- There are **five** general results that can be useful
  - If $y = (f(x))^n$ then $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$
  - If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - If $y = \ln(f(x))$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
  - If $y = \sin(f(x))$ then $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - If $y = \cos(f(x))$ then $\frac{dy}{dx} = -f'(x)\sin(f(x))$

## Worked example

a) Find the derivative of $y = (x^2 - 5x + 7)^7$.

STEP 1 Identify the two functions and rewrite

$y = u^7$ i.e. $f(u) = u^7$

$u = x^2 - 5x + 7$ i.e. $g(x) = x^2 - 5x + 7$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

$\frac{dy}{du} = 7u^6$ $\frac{du}{dx} = 2x - 5$

STEP 3 Apply chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$\frac{dy}{dx} = 7u^6(2x-5)$

and substitute u back for g(x)

$$\frac{dy}{dx} = 7(2x-5)(x^2-5x+7)^6$$

b) Find the derivative of $y = \sin(e^{2x})$.

$y = \sin(e^{2x})$ "... differentiate sin □, ignore $e^{2x}$..."

$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$ "... multiply by derivative of $e^{2x}$ ..."

" $y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$ "
or by applying chain rule again

$\therefore \frac{dy}{dx} = 2e^{2x}\cos(e^{2x})$

## Product Rule

### What is the product rule?

- The **product rule** states if $y$ is the product of two functions $u(x)$ and $v(x)$ then

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$$

- **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

### How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of $x$"
    - $\sin x \cos x$ is a product, "sin *x* times cos $x$"

### How do I use the product rule?

- Make it clear what $u$, $v$, $u'$ and $v'$ are
  - arranging them in a square can help
    - opposite diagonals match up

**STEP 1**

Identify the two functions, $u$ and $v$

Differentiate both $u$ and $v$ with respect to $x$ to find $u'$ and $v'$

**STEP 2**

Obtain $\frac{dy}{dx}$ by applying the product rule formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding $u'$ and $v'$

## Worked example

a) Find the derivative of $y = e^x \sin x$.

$y = e^x \sin x$

STEP 1 Identify functions and differentiate

$u = e^x$ $\quad v = \sin x$

$u' = e^x$ $\quad v' = \cos x$

arranging u, v, u', v' in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule: '$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$'

(As it is given in the formula booklet)

$y' = e^x \cos x + e^x \sin x$

$\therefore \frac{dy}{dx} = e^x(\cos x + \sin x)$

It is straightforward to take a factor of $e^x$ out

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$y = 5x^2 \cos 3x^2$

STEP 1 $\quad u = 5x^2$ $\quad v = \cos 3x^2$

$u' = 10x$ $\quad v' = -\sin 3x^2 \times 6x$ ← chain rule

$v' = -6x \sin 3x^2$

STEP 2 $\quad y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$\therefore \frac{dy}{dx} = 10x(\cos 3x^2 - 3x^2 \sin 3x^2)$

## Quotient Rule

### What is the quotient rule?

- The **quotient rule** states if $y$ is the quotient $\frac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

### How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of $x$
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

### How do I use the quotient rule?

- Make it clear what $u$, $v$, $u'$ and $v'$ are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

**STEP 1**

Identify the two functions, $u$ and $v$

Differentiate both $u$ and $v$ with respect to $x$ to find $u'$ and $v'$

**STEP 2**

Obtain $\frac{dy}{dx}$ by applying the quotient rule formula $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important

Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding $u'$ and $v'$,

### Worked example

Differentiate $f(x) = \dfrac{\cos 2x}{3x+2}$ with respect to $x$.

STEP 1 Identify u and v, differentiate

$u = \cos 2x$  $v = 3x + 2$

$u' = -2\sin 2x$  $v' = 3$

↑ chain rule  ↑ opposite diagonals match up

STEP 2 Apply quotient rule: '$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$'

(As it is given in the formula booklet)

$$f'(x) = \frac{(3x+2)(-2\sin 2x) - (\cos 2x)(3)}{(3x+2)^2}$$

$$\therefore f'(x) = \frac{-2(3x+2)\sin 2x - 3\cos 2x}{(3x+2)^2}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)

## 5.2.3 Related Rates of Change

## Related Rates of Change

### What is meant by rates of change?

- A **rate of change** is a measure of how a quantity is changing with respect to another quantity
- Mathematically rates of change are **derivatives**
  - $\frac{\mathrm{d}V}{\mathrm{d}r}$ could be the rate at which the volume of a sphere changes relative to how its radius is changing
- Context is important when interpreting positive and negative rates of change
  - A positive rate of change would indicate an increase
    - e.g. the change in volume of water as a bathtub fills
  - A negative rate of change would indicate a decrease
    - e.g. the change in volume of water in a leaking bucket

### What is meant by related rates of change?

- **Related rates of change** are connected by a linking variable or parameter
  - this is usually **time**, represented by $t$
  - **seconds** is the standard unit for time but this will depend on context
- e.g. Water running into a large bowl
  - both the height and volume of water in the bowl change with time
  - time is the linking parameter

### How do I solve problems involving related rates of change?

- Use of chain rule

$$y = g(u) \quad u = f(x) \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

- Chain rule is given in the **formula booklet** in the format above
  - **Different letters** may be used relative to the context
    - e.g. $V$ for **volume**, $S$ for **surface area**, $h$ for **height**, $r$ for **radius**
- Problems often involve one quantity being **constant**
  - so another quantity can be expressed in terms of a **single** variable
  - this makes finding a derivative a lot easier
- For **time** problems at least, it is more convenient to use

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}y}{\mathrm{d}x}$$

and if it is more convenient to find $\frac{\mathrm{d}x}{\mathrm{d}y}$ than $\frac{\mathrm{d}y}{\mathrm{d}x}$ then use chain rule in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}y}$$

- **Neither** of these alternative versions of chain rule are in the **formula booklet**

**STEP 1**
Write down the rate of change given and the rate of change required
(If unsure of the rates of change involved, use the units given as a clue
e.g. $\mathrm{m\,s^{-1}}$ (metres per second) would be the rate of change of length, per time, $\frac{\mathrm{d}l}{\mathrm{d}t}$)

**STEP 2**
Use chain rule to form an equation connecting these rates of change with a third rate
The third rate of change will come from a related quantity such as volume, surface area, perimeter

**STEP 3**
Write down the formula for the related quantity (volume, etc) accounting for any fixed quantities
Find the third rate of change of the related quantity (derivative) using differentiation

**STEP 4**
Substitute the derivative and known rate of change into the equation and solve it

## Worked example

A cuboid has a square cross-sectional area of side length $x$ cm and a fixed height of 5 cm.
The volume of the cuboid is increasing at a rate of 20 $cm^3\ s^{-1}$.
Find the rate at which the side length is increasing at the point when its side length is 3 cm.

STEP 1: Write down rates of change given and required

$\frac{dV}{dt} = 20$ (Units are $cm^3$ (volume) $s^{-1}$ (per second))

$\frac{dx}{dt}$ is required

STEP 2: Form equation from chain rule and a third 'connecting' rate

$$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$$

STEP 3: Formula for linking quantity, and its derivative

Volume (of a cuboid) is the link

$V = x^2 \times 5 = 5x^2$ (Cross-section is square, height is constant)

Differentiate, $\frac{dV}{dx} = 10x$

STEP 4: Substitute and solve

$$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$$

$$20 = \frac{dx}{dt} \times 10(3)$$

($x$, side length is 3)

$\therefore \frac{dx}{dt} = \frac{2}{3}$ $cm\ s^{-1}$

# 5.2.4 Second Order Derivatives

## Second Order Derivatives

### What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = f(x)$
  - $\frac{dy}{dx} = f'(x)$ (First order derivative)
  - $\frac{d^2y}{dx^2} = f''(x)$ (Second order derivative)
- Note the position of the superscript 2's
  - **d**ifferentiating twice (so $\mathbf{d^2}$) with respect to $x$ twice (so $x^2$)
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate** of **change** of a function
  - a **second order derivative** is the **rate** of **change** of the **rate** of **change** of a function
    - i.e. the **rate** of **change** of the function's **gradient**
- **Second order derivatives** can be used to
  - test for local minimum and maximum points
  - help determine the nature of stationary points
  - determine the concavity of a function
  - graph derivatives

### How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions, roots,** etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule

## Worked example

Given that $f(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a) Find $f'(x)$ and $f''(x)$.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ ← REWRITE AS POWERS OF x

$f'(x) = -(\frac{1}{2})x^{\frac{1}{2}-1} + 3(-\frac{1}{2})x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$ ← DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}(-\frac{1}{2})x^{-\frac{1}{2}-1} - \frac{3}{2}(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$ ← DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b) Evaluate $f''(3)$.

Give your answer in the form $a\sqrt{b}$, where $b$ is an integer and $a$ is a rational number.

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$ ← $x^{\frac{3}{2}} = x\sqrt{x}$ $\quad x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$ ← RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$

## 5.2.5 Further Applications of Differentiation

### Stationary Points & Turning Points

**What is the difference between a stationary point and a turning point?**

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** is a type of stationary point, but in addition the **function changes** from **increasing** to **decreasing**, or **vice versa**
  - The curve **'turns'** from **'going upwards'** to **'going downwards'** or **vice versa**
  - **Turning points** will either be **(local) minimum** or **maximum** points
- A **point** of **inflection** ***could*** also be a **stationary point** but is **not** a turning point

**How do I find stationary points and turning points?**

- For the function $y = f(x)$, **stationary points** can be found using the following process

  **STEP 1**

  Find the **gradient function,** $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$

  **STEP 2**

  Solve the equation $f'(x) = 0$ to find the $x$-coordiante(s) of any stationary points

  **STEP 3**

  If the $y$-coordinates of the stationary points are also required then substitute the $x$-coordinate(s) into $f(x)$

- A GDC will solve $f'(x) = 0$ and most will find the coordinates of turning points (minimum and maximum points) in graphing mode

## Testing for Local Minimum & Maximum Points

### What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
  - i.e. $f'(x) = 0$
- A **local minimum** point, $(x,\ f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of $x$
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x,\ f(x))$ will be the highest value of $f(x)$ in the **local** vicinity of the value of $x$
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend** to **infinity** for **large** values of $x$ (and/or **minus infinity** for **large negative** values of $x$)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of $f(x)$ for **all values** of $x$
  - similar for a **global** maximum point

### How do I find local minimum & maximum points?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function $f(x)$ ...

  **STEP 1**

  Find $f'(x)$ and solve $f'(x) = 0$ to find the $x$-coordinates of any stationary points

  **STEP 2** (Second derivative)

  Find $f''(x)$ and evaluate it at each of the stationary points found in **STEP 1**

  **STEP 3** (Second derivative)

  - If $f''(x) = 0$ then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
  - If $f''(x) > 0$ then the curve of the graph of $y = f(x)$ is **concave up** and the stationary point is a **local minimum** point
  - If $f''(x) < 0$ then the curve of the graph of $y = f(x)$ is **concave down** and the stationary point is a **local maximum** point

  **STEP 4** (First derivative)

  Find the sign of the first derivative just either side of the stationary point; i.e. evaluate $f'(x-h)$ and $f'(x+h)$ for small $h$

- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h)<0,\ f'(x)=0,\ f'(x+h)>0$
- A **local maximum point** changes the **function** from **increasing** to **decreasing**
  - the **gradient** changes from **positive** to **negative**
  - $f'(x-h)>0,\ f'(x)=0,\ f'(x+h)<0$

- A **stationary point** of **inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not change** sign
  - $f'(x-h)>0,\ f'(x+h)>0$ or $f'(x-h)<0,\ f'(x+h)<0$
  - a **point** of **inflection** does **not** necessarily have $f'(x)=0$
    - this method will only find those that do - and are often called **horizontal points** of **inflection**

### Worked example

Find the coordinates and the nature of any stationary points on the graph of $y = f(x)$ where $f(x) = 2x^3 - 3x^2 - 36x + 25$.

At stationary points, $f'(x) = 0$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$6(x^2 - x - 6) = 0$

$(x-3)(x+2) = 0$

$x = 3, \quad y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$

$x = -2, \quad y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$

Using the second derivative to determine their nature

$f''(x) = 12x - 6 = 6(2x - 1)$

$f''(3) = 6(2 \times 3 - 1) = 30 > 0$

$\therefore x = 3$ is a local minimum point

$f''(-2) = 6(2 \times -2 - 1) = -30 < 0$

$\therefore x = -2$ is a local maximum point

(Note: In this case, both stationary points are turning points)

**Turning points are:**

**$(3, -56)$ local minimum point**

**$(-2, 69)$ local maximum point**

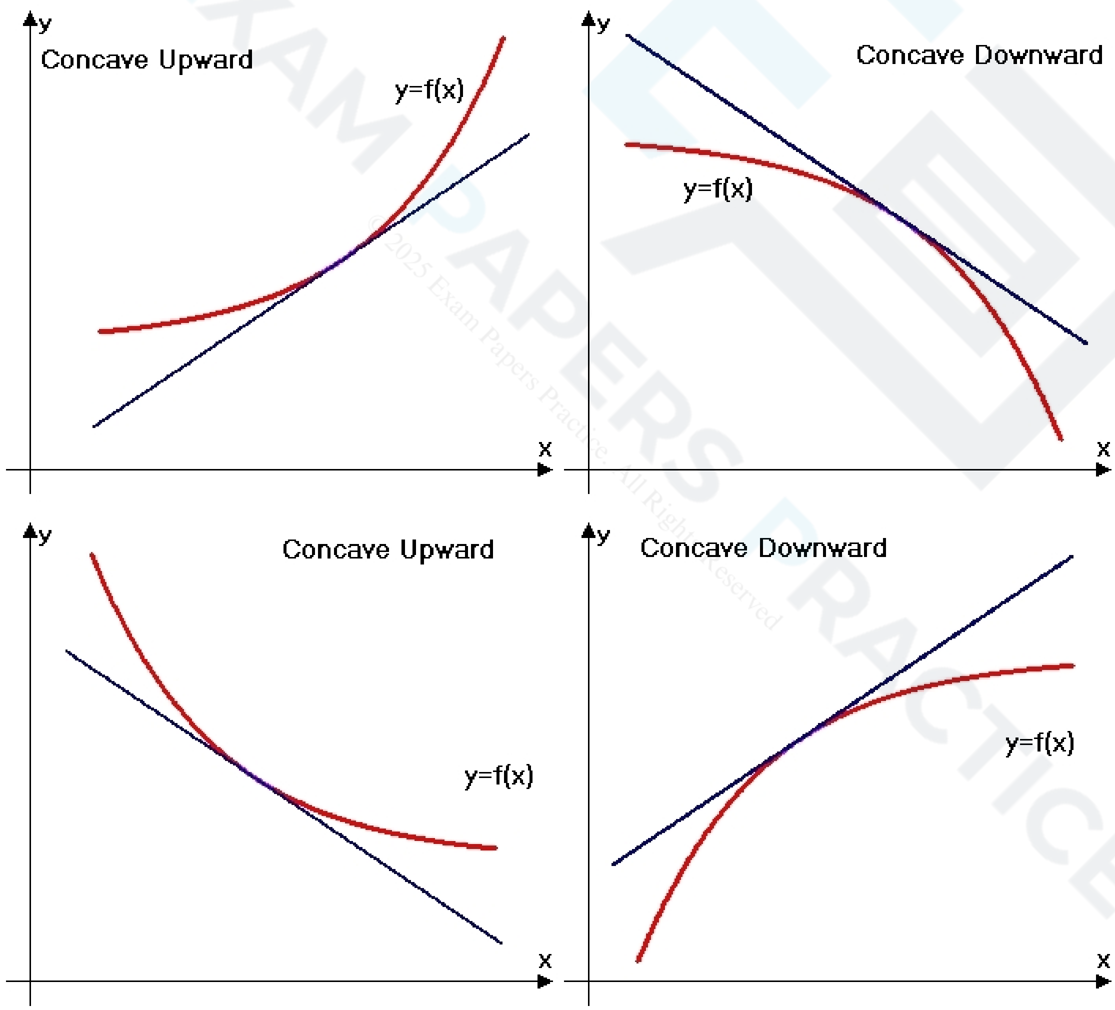
Use a GDC to graph $y = f(x)$ and the max/min solving feature to check the answers.

# 5.2.6 Concavity & Points of Inflection

## Concavity of a Function

**What is concavity?**

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if $f''(x) < 0$ for all values of $x$ in an interval
  - a curve is **CONCAVE UP** if $f''(x) > 0$ for all values of $x$ in an interval



## Worked example

The function $f(x)$ is given by $f(x) = x^3 - 3x + 2$.

a) Determine whether the curve of the graph of $y = f(x)$ is concave down or concave up at the points where $x = -2$ and $x = 2$.

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

$f''(-2) = 6 \times -2 = -12 < 0$ (concave down)

$f''(2) = 6 \times 2 = 12 > 0$ (concave up)

**At $x = -2$, $y = f(x)$ is concave down**
**At $x = 2$, $y = f(x)$ is concave up**

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answers are sensible

b) Find the values of $x$ for which the curve of the graph $y = f(x)$ of is concave up.

$f''(x) = 6x$ from part (a)

Concave up is $f''(x) > 0$

$6x > 0$ when $x > 0$

**$\therefore$ $y = f(x)$ is concave up for $x > 0$**

Use your GDC to check your answer

## Points of Inflection

**What is a point of inflection?**

- A point at which the curve of the graph of $y = f(x)$ changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion,** may sometimes be used

**What are the conditions for a point of inflection?**

- A point of inflection requires **BOTH** of the following two conditions to hold
  - the **second derivative** is zero
    - $f''(x) = 0$

  **AND**

  - the graph of $y = f(x)$ changes **concavity**
    - $f''(x)$ changes **sign** through a **point** of **inflection**

- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
  - points where $f''(x) = 0$ could be **local minimum** or **maximum** points
    - the **first derivative** test would be needed
  - However, if it is already known $f(x)$ has a point of inflection at $x = a$, say, then $f''(a) = 0$

**What about the first derivative, like with turning points?**

- A **point** of **inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ( $f'(x) = 0$ )
  - If it does, it is also a **stationary point** and is often called a **horizontal point** of **inflection**
    - the tangent to the curve at this point would be horizontal
  - The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

**How do I find the coordinates of a point of inflection?**

- For the function $f(x)$

  **STEP 1**

  Differentiate $f(x)$ **twice** to find $f''(x)$ and solve $f''(x) = 0$ to find the $x$-coordinates of possible points of inflection

  **STEP 2**

  Use the **second derivative** to **test** the **concavity** of $f(x)$ either side of $x = a$

  - If $f''(x) < 0$ then $f(x)$ is concave down
  - If $f''(x) > 0$ then $f(x)$ is concave up

  If concavity changes, $x = a$ is a **point of inflection**

  **STEP 3**

  If required, the $y$-coordinate of a point of inflection can be found by substituting the $x$-coordinate into $f(x)$

## Worked example

Find the coordinates of the point of inflection on the graph of $y = 2x^3 - 18x^2 + 24x + 5$.
Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve $f''(x) = 0$

$f(x) = 2x^3 - 18x^2 + 24x + 5$

$f'(x) = 6x^2 - 36x + 24$

$f''(x) = 12x - 36$

$12x - 36 = 0$ when $x = 3$

STEP 2: Use the second derivative to test concavity

$f''(3) = 0$

$f''(2.9) < 0$ (concave down)

$f''(3.1) > 0$ (concave up)

$\therefore$ concavity changes through $x = 3$

STEP 3: The y-coordinate is required

$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

Since $f''(3) = 0$ AND the graph of $y = f(x)$ changes concavity through $x = 3$, the point $(3, -31)$ is a point of inflection.

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answer is sensible