



5.2 Further Differentiation

Contents

- ✤ 5.2.1 Differentiating Special Functions
- ✤ 5.2.2 Techniques of Differentiation
- ✤ 5.2.3 Second Order Derivatives
- ✤ 5.2.4 Further Applications of Differentiation
- * 5.2.5 Concavity & Points of Inflection
- ★ 5.2.6 Derivatives & Graphs



5.2.1 Differentiating Special Functions

Differentiating Trig Functions

How do I differentiate sin and cos?

- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$
- For the linear function ax + b, where a and b are constants,
 - the derivative of $y = \sin(ax + b)$ is $\frac{dy}{dx} = a\cos(ax + b)$
 - the derivative of $y = \cos(ax + b)$ is $\frac{dy}{dx} = -a\sin(ax + b)$
- For the general function f(x),

• the derivative of
$$y = \sin(f(x))$$
 is $\frac{dy}{dx} = f'(x)\cos(f(x))$

- the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
- These last two sets of results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in radians
 - Ensure you know how to change the angle mode on your GDC



Worked example

a) Find f'(x) for the functions

- $f(x) = \sin x$
- $f(x) = \cos 2x$
- iii. $f(x) = 3\sin 4x \cos(2x 3)$
 - i. f'(∞) = Co9 ∞

iii. Differentiate 'term by term' $f'(x) = 3(h\cos hx) - (-2\sin (2x-3))$ $f'(x) = 12\cos hx + 2\sin (2x-3)$

b)

Find the gradient of the tangent to the curve $y = \sin\left(2x + \frac{\pi}{6}\right)$ at the point where

$$x = \frac{\pi}{8}$$

Give your answer as an exact value.

Gradient of tangent is equal to gradient of curve

$$\frac{dy}{dx} = 2\cos(2x + \frac{\pi}{6})$$
At $x = \frac{\pi}{8}$. $\frac{dy}{dx} = 2\cos(2(\frac{\pi}{8}) + \frac{\pi}{6})$ Ensure your GDC is
in RADIANS mode
: Gradient of tangent, at $x = \frac{\pi}{8}$, is $\frac{1}{2}(\sqrt{6} - \sqrt{2})$



Differentiating e^x & Inx

How do I differentiate exponentials and logarithms?

• The derivative of
$$y = e^x$$
 is $\frac{dy}{dx} = e^x$ where $x \in \mathbb{R}$

• The derivative of
$$y = \ln x$$
 is $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$

- For the **linear** function ax + b, where a and b are constants,
 - the derivative of $y = e^{(ax+b)}$ is $\frac{dy}{dx} = ae^{(ax+b)}$
 - the derivative of $y = \ln(ax + b)$ is $\frac{dy}{dx} = \frac{a}{(ax + b)}$
 - in the special case b=0, $\frac{dy}{dx} = \frac{1}{x}$ (*a*'s cancel)
- For the general function f(x),

• the derivative of
$$y = e^{f(x)}$$
 is $\frac{dy}{dx} = f'(x)e^{f(x)}$

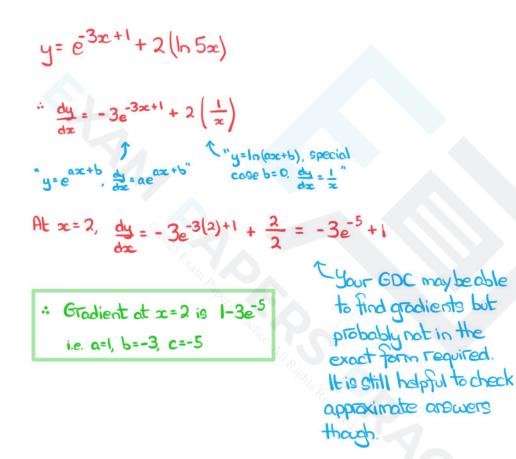
- the derivative of $y = \ln(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
- The last two sets of results can be derived using the **chain rule**



Worked example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where x = 2 gving your answer in the form $y = a + be^{c}$, where a, b and c are integers to be found.





5.2.2 Techniques of Differentiation

Chain Rule

What is the chain rule?

• The chain rule states if \boldsymbol{y} is a function of \boldsymbol{u} and \boldsymbol{u} is a function of \boldsymbol{X} then y = f(u(x))

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

- This is given in the formula booklet
- In function notation this could be written

$$y = f(g(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x))g'(x)$$

How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
 - "function of a function"
 - these can be identified as the variable (usually X) does not 'appear alone'
 - Sin X not a composite function, X 'appears alone'
 - sin(3x+2) is a composite function; X is tripled and has 2 added to it before the sine function is applied

How do I use the chain rule?

STEP 1

Identify the two functions

Rewrite *y* as a function of *u*; y = f(u)Write *u* as a function of *x*; u = g(x)

STEP 2

Differentiate *y* with respect to *u* to get $\frac{dy}{du}$ Differentiate *u* with respect to *x* to get $\frac{du}{dx}$

STEP 3



Obtain
$$\frac{dy}{dx}$$
 by applying the formula $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and substitute u back in for $g(x)$

• In trickier problems chain rule may have to be applied more than once

Are there any standard results for using chain rule?

• There are **five** general results that can be useful

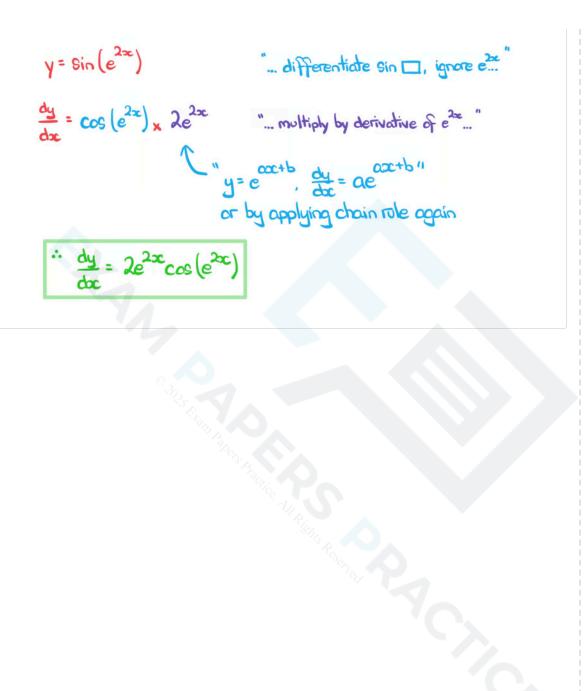
If
$$y = (f(x))^n$$
 then $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$
If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$
If $y = \ln(f(x))$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
If $y = \sin(f(x))$ then $\frac{dy}{dx} = f'(x)\cos(f(x))$
If $y = \cos(f(x))$ then $\frac{dy}{dx} = -f'(x)\sin(f(x))$



Worked example a) Find the derivative of $y = (x^2 - 5x + 7)^7$. STEP 1 Identify the two functions and rewrite $y = v^7$ i.e. $f(v) = v^7$ $v = x^2 - 5x + 7$ i.e. $g(x) = x^2 - 5x + 7$ STEP 2 Find dy and dv dv = 2x - 5STEP 3 Apply chain role, $dy = dv \times dv$ $dx = dv \times dx$ Chain rule is in the formula booklet $dy = 7v^6 (2x - 5)$ and substitute v back for g(x) $dy = 7(2x - 5)(x^2 - 5x + 7)^6$

b) Find the derivative of $y = \sin(e^{2x})$.







Product Rule

What is the product rule?

• The product rule states if Y is the product of two functions U(X) and V(X) then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

y = uv

- This is given in the formula booklet
- In function notation this could be written as

$$y = f(x)g(x)$$
$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$$

• **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$
$$y' = uv' + vu'$$

Final answers should match the notation used throughout the question

How do I know when to use the product rule?

- The product rule is used when we are trying to differentiate the product of two functions
 - these can easily be confused with composite functions (see **chain rule**)
 - sin(cos x) is a composite function, "sin of cos of X"
 - sin xcos x is a product, "sin x times cos x"

How do I use the product rule?

- Make it clear what u, v, u' and v' are
 - arranging them in a square can help
 - opposite diagonals match up

STEP 1

Identify the two functions, $\boldsymbol{\mathit{U}}$ and $\boldsymbol{\mathit{V}}$

Differentiate both u and v with respect to x to find u' and v'

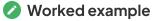
STEP 2

Obtain $\frac{dy}{dx}$ by applying the product rule formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Simplify the answer if straightforward to do so or if the question requires a particular form

• In trickier problems **chain rule** may have to be used when finding u' and v'





a) Find the derivative of $y = e^{X} \sin x$.

y = e^x Sin x STEP 1 Identify functions and differentiate $u = e^{x}$ v = Sin x $u' = e^{x}$ v' = cos x 1 arranging u, v, u', v' in a square makes product role 'diagonal pairs' STEP 2 Apply product role: 'dy = udv + vdu' dx dx dx' (As it is given in the formula booklet)

$$y' = e^{x} \cos x + e^{x} \sin x$$

$$\frac{dy}{dx} = e^{x} (\cos x + \sin x)$$

It is straightforward to take a factor of e aut

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$$y = 5x^{2} \cos 3x^{2}$$

STEP 1 $u = 5x^{2}$ $v = \cos 3x^{2}$ chain rule
 $u' = 10x$ $v' = -9in 3x^{2} x 6x$
 $v' = -6x \sin 3x^{2}$
STEP 2 $y' = -30x^{3} \sin 3x^{2} + 10x \cos 3x^{2}$
 $\therefore \frac{dy}{dx} = 10x(\cos 3x^{2} - 3x^{2} \sin 3x^{2})$



Quotient Rule

What is the quotient rule?

• The **quotient rule** states if y is the quotient $\frac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

- This is given in the formula booklet
- In function notation this could be written

$$y = \frac{f(x)}{g(x)}$$
$$\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

• As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$
$$y' = \frac{vu' - uv'}{v^2}$$

• Final answers should match the notation used throughout the question

How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and
 - denominator are functions of \boldsymbol{X}
 - if the **numerator** is a **constant**, **negative powers** can be used
 - if the **denominator** is a **constant**, treat it as a **factor** of the expression

How do I use the quotient rule?

- Make it clear what u, v, u' and v' are
 - arranging them in a square can help
 - opposite diagonals match up (like they do for product rule)

STEP 1

Identify the two functions, ${\it U}$ and ${\it V}$



Differentiate both u and v with respect to x to find u' and v'

STEP 2

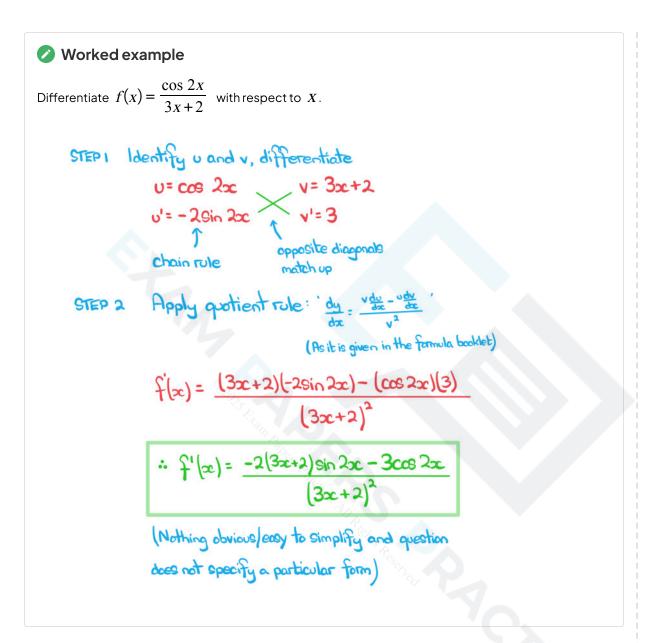
Obtain
$$\frac{dy}{dx}$$
 by applying the quotient rule formula $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important

Simplify the answer if straightforward or if the question requires a particular form

• In trickier problems **chain rule** may have to be used when finding u' and v',







5.2.3 Second Order Derivatives

Second Order Derivatives

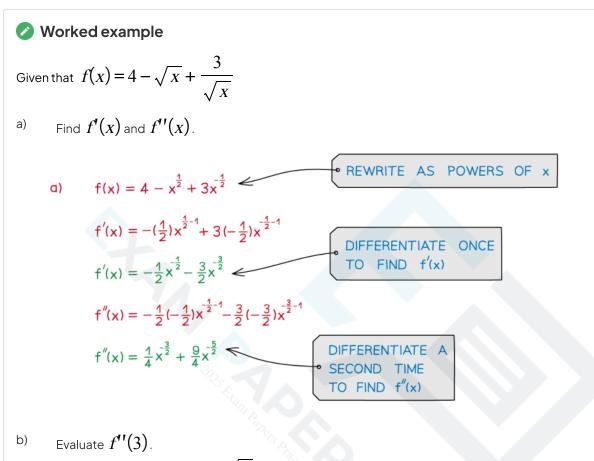
What is the second order derivative of a function?

- If you differentiate the derivative of a function (i.e. differentiate the function a second time) you get the second order derivative of the function
- There are two forms of notation for the second order derivative
 - y = f(x)
 - $\frac{dy}{dx} = f'(x)$ (First order derivative) $\frac{d^2v}{dx}$
 - $\frac{d^2y}{dx^2} = f''(x)$ (Second order derivative)
- Note the position of the superscript 2's
 - differentiating twice (so d^2) with respect to X twice (so x^2)
- The second order derivative can be referred to simply as the second derivative
 - Similarly, the first order derivative can be just the first derivative
- A first order derivative is the rate of change of a function
 - a second order derivative is the rate of change of the rate of change of a function
 - i.e. the rate of change of the function's gradient
- Second order derivatives can be used to
 - test for local minimum and maximum points
 - help determine the nature of stationary points
 - determine the concavity of a function
 - graph derivatives

How do I find a second order derivative of a function?

- By differentiating twice!
- This may involve
 - rewriting fractions, roots, etc as negative and/or fractional powers
 - differentiating trigonometric functions, exponentials and logarithms
 - using chain rule
 - using product or quotient rule





Give your answer in the form $a\sqrt{b}$, where b is an integer and a is a rational number.

b)
$$f''(x) = \frac{1}{4x\sqrt{x}} + \frac{9}{4x^2\sqrt{x}}$$

 $x^2 = x\sqrt{x}$ $x^2 = x^2\sqrt{x}$
 $f''(3) = \frac{1}{12\sqrt{3}} + \frac{9}{36\sqrt{3}}$
 $= \frac{12}{36\sqrt{3}} = \frac{1}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{9}$
 $f''(3) = \frac{1}{9}\sqrt{3}$
RATIONALISE
DENOMINATOR



5.2.4 Further Applications of Differentiation

Stationary Points & Turning Points

What is the difference between a stationary point and a turning point?

- A stationary point is a point at which the gradient function is equal to zero
 - The tangent to the curve of the function is horizontal
- A turning point is a type of stationary point, but in addition the function changes from increasing to decreasing, or vice versa
 - The curve 'turns' from 'going upwards' to 'going downwards' or vice versa
 - Turning points will either be (local) minimum or maximum points
- A point of inflection could also be a stationary point but is not a turning point

How do I find stationary points and turning points?

• For the function y = f(x), stationary points can be found using the following process

STEP 1

Find the gradient function, $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$

STEP 2

Solve the equation f'(x) = 0 to find the x-coordiante(s) of any stationary points

STEP 3

If the *Y*-coordinates of the stationary points are also required then substitute the *X*-coordinate(s) into f(x)

• A GDC will solve f'(x) = 0 and most will find the coordinates of turning points (minimum and maximum points) in graphing mode



Testing for Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local minimum and maximum points are two types of stationary point
 - The gradient function (derivative) at such points equals zero
 - i.e. f'(x) = 0
- A local minimum point, (X, f(X)) will be the lowest value of f(X) in the local vicinity of the value of X
 - The function may reach a **lower** value further afield
- Similarly, a local maximum point, (X, f(X)) will be the highest value of f(X) in the local vicinity of the value of X
 - The function may reach a greater value further afield
- The graphs of many functions tend to infinity for large values of X (and/or minus infinity for large negative values of X)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A global minimum point would represent the lowest value of f(x) for all values of X
 - similar for a **global** maximum point

How do I find local minimum & maximum points?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
 - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function f(x) ...

STEP 1

Find f'(x) and solve f'(x) = 0 to find the *x*-coordinates of any stationary points

STEP 2 (Second derivative)

Find f''(x) and evaluate it at each of the stationary points found in STEP 1

STEP 3 (Second derivative)

- If f''(x) = 0 then the nature of the stationary point cannot be determined; use the first derivative method (STEP 4)
- If f''(x) > 0 then the curve of the graph of y = f(x) is **concave up** and the stationary point is a **local minimum** point
- If f''(x) < 0 then the curve of the graph of y = f(x) is **concave down** and the stationary point is a **local maximum** point

STEP 4 (First derivative)

Find the sign of the first derivative just either side of the stationary point; i.e. evaluate f'(x - h) and f'(x + h) for small h



- A local minimum point changes the function from decreasing to increasing
 - the gradient changes from negative to positive

$$f'(x-h) < 0, f'(x) = 0, f'(x+h) > 0$$

- A local maximum point changes the function from increasing to decreasing
 - the gradient changes from positive to negative

$$f'(x-h) > 0, \ f'(x) = 0, \ f'(x+h) < 0$$

- A stationary point of inflection results from the function either increasing or decreasing on both sides of the stationary point
 - the gradient does not change sign
 - f'(x-h) > 0, f'(x+h) > 0 or f'(x-h) < 0, f'(x+h) < 0
 - a point of inflection does not necessarily have f'(x) = 0
 - this method will only find those that do and are often called horizontal points of inflection

Worked example

Find the coordinates and the nature of any stationary points on the graph of y = f(x) where $f(x) = 2x^3 - 3x^2 - 36x + 25$.



At stationary points, f'(x)=0 $f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$ $6(x^2 - x - 6) = 0$ $(\infty-3)(\infty+2)=0$ x = 3, $y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$ x=-2, $y=f(-2)=2(-2)^3-3(-2)^2-36(-2)+25=69$ Using the second derivative to determine their nature f''(x) = 12x - 6 = 6(2x - 1)f''(3) = 6(2x3-1) = 30 > 0: c=3 is a local minimum point f''(-2) = 6(2x-2-1) = -30 < 0* x=-2 is a local maximum point (Note: In this case, both stationary points are turning points) Turning points are: (3,-56) local minimum point (-2, 69) local maximum point Use a GDC to graph y=f(x) and the max min solving feature to check the answers.

For more help, please visit www.exampaperspractice.co.uk



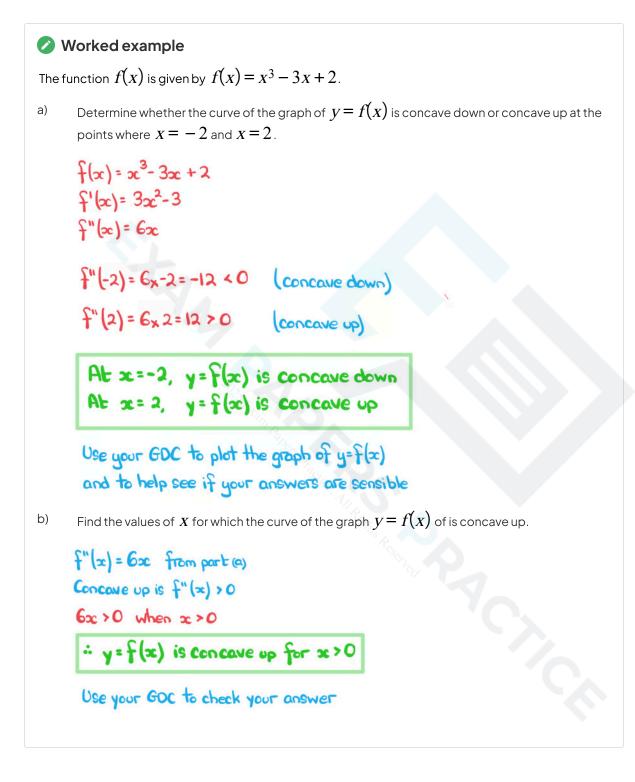
5.2.5 Concavity & Points of Inflection

Concavity of a Function

What is concavity?

- Concavity is the way in which a curve (or surface) bends
- Mathematically,
 - a curve is **CONCAVE DOWN** if f''(x) < 0 for all values of x in an interval
 - a curve is **CONCAVE UP** if f''(x) > 0 for all values of X in an interval







Points of Inflection

What is a point of inflection?

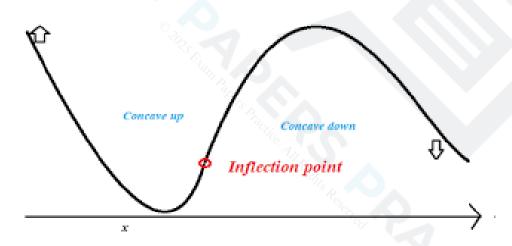
- A point at which the curve of the graph of y = f(x) changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion**, may sometimes be used

What are the conditions for a point of inflection?

- A point of inflection requires **BOTH** of the following two conditions to hold
 - the **second derivative** is zero
 - f''(x) = 0

AND

- the graph of y = f(x) changes **concavity**
 - f''(x) changes sign through a point of inflection



- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
 - points where f''(x) = 0 could be local minimum or maximum points
 the first derivative test would be needed
 - However, if it is already known f(x) has a point of inflection at x = a, say, then f''(a) = 0

What about the first derivative, like with turning points?



- A **point** of **inflection**, unlike a turning point, does not necessarily have to have a first derivative value of O(f'(x) = 0)
 - If it does, it is also a stationary point and is often called a horizontal point of inflection
 the tangent to the curve at this point would be horizontal
 - The **normal distribution** is an example of a commonly used function that has a graph with two nonstationary points of inflection

How do I find the coordinates of a point of inflection?

• For the function f(x)

STEP 1

Differentiate f(x) twice to find f''(x) and solve f''(x) = 0 to find the *x*-coordinates of possible points of inflection

STEP 2

Use the second derivative to test the concavity of f(x) either side of x = a

- If f''(x) < 0 then f(x) is concave down
- If f''(x) > 0 then f(x) is concave up

If concavity changes, x = a is a point of inflection

STEP 3

If required, the *Y*-coordinate of a point of inflection can be found by substituting the *X*-coordinate into f(x)





Find the coordinates of the point of inflection on the graph of $y = 2x^3 - 18x^2 + 24x + 5$. Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve f''(x) = 0 $f(x) = 2x^3 - 18x^2 + 24x + 5$ $f'(x) = 6x^2 - 36x + 24$ f''(x) = 12x - 3612x - 36 = 0 when x = 3

STEP 2: Use the second derivative to test concavity f"(3)=0 f"(2.9) < 0 (concave down) f"(3.1)>0 (concave up) .: concavity changes through x=3

STEP 3: The y-coordinate is required $f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

> Since f"(3)=0 <u>AND</u> the graph of y=f(x) changes concavity through x=3, the point (3,-31) is a point of inflection.

Use your GDC to plot the graph of y=f(x)and to help see if your answer is sensible



5.2.6 Derivatives & Graphs

Derivatives & Graphs

How are derivatives and graphs connected?

- If the graph of a function y = f(x) is known, or can be sketched, then it is also possible to sketch the graphs of the derivatives y = f'(x) and y = f''(x)
- The key properties of a graph include
 - the y-axis intercept
 - the **X**-axis intercepts the roots of the function; where f(x) = 0
 - stationary points; where f'(x) = 0
 - turning points (local) **minimum** and **maximum** points
 - (horizontal) **points of inflection**
 - (non-stationary, $f'(x) \neq 0$) points of inflection
 - asymptotes vertical and horizontal
 - intervals where the graph is increasing and decreasing
 - intervals where the graph is concave down and concave up
- Not all graphs have all of these properties and **not** all can be determined without knowing the expression of the function
- However questions will provide enough information to sketch
 - the **shape** of the graph
 - some of the key properties such as roots or turning points

How do I sketch the graph of y = f'(x) from the graph of y = f(x)?

- The graph of y = f'(x) will have its
 - X-axis intercepts at the X-coordinates of the stationary points of y = f(x)
 - turning points at the X-coordinates of the points of inflection of y = f(x)
- For intervals where y = f(x) is concave up, y = f'(x) will be increasing
- For intervals where y = f(x) is concave down, y = f'(x) will be decreasing
- For intervals where y = f(x) is increasing, y = f'(x) will be positive
- For intervals where y = f(x) is decreasing, y = f'(x) will be negative

How do I sketch the graph of y = f''(x) from the graph of y = f(x)?

- First sketch the graph of y = f'(x) from y = f(x), as per the above process
- Then, using the same process, sketch the graph of y = f''(x) from the graph of y = f'(x)
- There are a couple of things you can deduce about the graph of y = f''(x) directly from the graph of y = f(x)



- The graph of y = f''(x) will have its X-axis intercepts at the X-coordinates of the points of inflection of V = f(x)
- For intervals where y = f(x) is concave up, y = f''(x) will be positive
- For intervals where y = f(x) is concave down, y = f''(x) will be negative

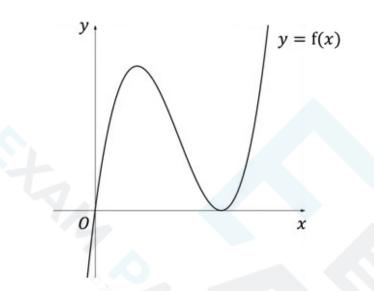
Is it possible to sketch the graph of y = f(x) from the graph of a derivative?

- It is possible to sketch a graph of Y = f(x) by considering the reverse of the above
 - For intervals where y = f'(x) is **positive**, y = f(x) will be **increasing** but is **not** necessarily positive
 - For intervals where y = f'(x) is **negative**, y = f(x) will be **decreasing** but is **not** necessarily negative
 - Roots of y = f'(x) give the *X*-coordinates of the stationary points of y = f(x)
- There are some properties of the graph of y = f(x) that **cannot** be determined from the graph of V = f'(X)
 - the *Y***-axis intercept**
 - the intervals for which y = f(x) is positive and negative
 - the roots of V = f(X)
- Unless a specific point the curve passes through is known, the constant of integration cannot be determined
 - the exact location of the curve will remain unknown
 - but it will still be possible to sketch its shape
- If starting from the graph of the second derivative, y = f''(x), it is easier to sketch the graph of
 - y = f'(x) first, then sketch y = f(x)



Worked example

The graph of y = f(x) is shown in the diagram below.



On separate diagrams sketch the graphs of y = f'(x) and y = f''(x), labelling any roots and turning points.



