5.2 Integration

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Course	DP IB Maths
Section	5. Calculus
Topic	5.2 Integration
Difficulty	Medium

To be used by all students preparing for DP IB Maths AI SL
Students of other boards may also find this useful

Question 1

a) Substitute x-coordinate into $f'(x)$

$$f'(4) = 5(4) - 2 = \boxed{18}$$

b) We want the line through $(4, 2)$ with gradient 18.

Use $y - y_1 = m(x - x_1)$

$$y - 2 = 18(x - 4)$$

$$y - 2 = 18x - 72$$

$$\boxed{y = 18x - 70}$$

Powers of x integration formulae

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \int k \, dx = kx + c$$

$(n \neq -1)$ (k is a constant)

c) $f(x) = \int f'(x)\,dx$

constant of integration

$$f(x) = \int (5x-2)\,dx = \frac{5}{2}x^2 - 2x + c$$

But $y = f(x)$ goes through $(4, 2)$,
so $f(4) = 2$.

$$\frac{5}{2}(4)^2 - 2(4) + c = 2$$

$$\frac{5}{2}(16) - 2(4) + c = 2$$

$$40 - 8 + c = 2$$

$$32 + c = 2 \implies c = -30$$

$$\boxed{y = f(x) = \frac{5}{2}x^2 - 2x - 30}$$

Question 2

a) Substitute x-coordinate into $f'(x)$

$$f'(3) = -2(3)^2 + 11 = -18 + 11 = \boxed{-7}$$

b) We want the line through $(3, 8)$ with gradient -7.

Use $y - y_1 = m(x - x_1)$

$$y - 8 = -7(x - 3)$$

$$y - 8 = -7x + 21$$

$$\boxed{y = -7x + 29}$$

Powers of x integration formulae

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int k \, dx = kx + c \qquad (k \text{ is a constant})$$

c) $f(x) = \int f'(x) \, dx$

constant of integration

$$\int (-2x^2 + 11) \, dx = -\frac{2}{3}x^3 + 11x + c$$

But $y = f(x)$ goes through $(3, 8)$,
so $f(3) = 8$.

$$-\frac{2}{3}(3)^3 + 11(3) + c = 8$$

$$-\frac{2}{3}(27) + 11(3) + c = 8$$

$$-18 + 33 + c = 8$$

$$15 + c = 8 \Rightarrow c = -7$$

$$\boxed{y = f(x) = -\frac{2}{3}x^3 + 11x - 7}$$

Question 3

Trapezoidal Rule

$$\int_a^b y \, dx \approx \frac{1}{2}h\left((y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right)$$

where $h = \frac{b-a}{n}$

a) Use the Trapezoidal Rule

$$\int_0^1 y\,dx \approx \frac{1}{2}\left(\frac{1-0}{4}\right)\left((1+9)+2(2.25+4+6.25)\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}\right)(35) = \frac{35}{8}$$

$\boxed{\text{Area} \approx \frac{35}{8} \text{ units}^2}$ $\boxed{= 4.375 \text{ units}^2}$

Either form will get the marks!

b) $\int_0^1 (2x+1)^2\,dx = \frac{13}{3}$ from GDC

$\boxed{\text{Area} = \frac{13}{3} \text{ units}^2}$

Percentage error

$$\varepsilon = \left|\frac{V_A - V_E}{V_E}\right| \times 100\%$$

V_A is the approximate value

V_E is the exact value

c) $\varepsilon = \left| \dfrac{\frac{35}{8} - \frac{13}{3}}{\frac{13}{3}} \right| \times 100 = \dfrac{25}{26} = 0.961538\ldots$

$$\boxed{\varepsilon = 0.96\,\% \ (2\text{ d.p.})}$$

The approximation is inexact because it is based on adding up the areas of straight-sided trapezoids. The actual area is bounded on top by a curve (in this case a parabola).

Question 4

a) (i) $\boxed{P \text{ is the point } (3, 0)}$

(ii) By symmetry, V's x-coordinate is halfway between O's and P's x-coordinates.

$$\boxed{V \text{ is the point } (1.5, 4.5)}$$

(iii) The curve goes through point P(3, 0), so $y = 0$ when $x = 3$.

$$3(a(3) + 6) = 0$$

$$9a + 18 = 0$$

$$9a = -18 \implies \boxed{a = -2}$$

b) Area $= \int_0^3 x(-2x+6)\,dx = 9$ from GDC

$$\boxed{9\ m^2}$$

Question 5

a) For $f(x) = ax^2 + bx + c$, the axis of symmetry is $x = -\frac{b}{2a}$

$$-\frac{p}{2(-4)} = 2.5 \implies \frac{p}{8} = 2.5 \implies \boxed{p = 20}$$

The curve goes through $(2.5, 6)$, so $y = 6$ when $x = 2.5$.

$$-4(2.5)^2 + 20(2.5) + q = 6$$

$$-25 + 50 + q = 6$$

$$q + 25 = 6 \implies \boxed{q = -19}$$

b) The x-coordinates of A and B are the solutions to:

$$-4x^2 + 20x - 19 = 0$$

$$\Rightarrow \quad x = \frac{5-\sqrt{6}}{2} \quad \text{or} \quad x = \frac{5+\sqrt{6}}{2} \quad \text{from GDC}$$

$$= 1.275\ldots \qquad\qquad = 3.274\ldots$$

$$\boxed{\begin{array}{l} A \text{ is } \left(\frac{5-\sqrt{6}}{2}, 0\right) \\ B \text{ is } \left(\frac{5+\sqrt{6}}{2}, 0\right) \end{array}}$$

c) $$\text{Area of } R = \int_{\frac{5-\sqrt{6}}{2}}^{\frac{5+\sqrt{6}}{2}} \left(-4x^2 + 20x - 19\right) dx$$

$$= 4\sqrt{6} = 9.797958\ldots$$

from GDC

$$\boxed{\begin{array}{l} \text{Area of } R = 4\sqrt{6} \text{ units}^2 \\ \qquad\qquad\quad = 9.80 \text{ units}^2 \text{ (3 s.f.)} \end{array}}$$

Question 6

a) $f(x) = 0$ when $(5-2x) = 0$ or $(2+3x) = 0$

$2 + 3x = 0 \Rightarrow 3x = -2 \Rightarrow x = -\frac{2}{3}$

$5 - 2x = 0 \Rightarrow 2x = 5 \Rightarrow x = \frac{5}{2}$

$$\boxed{\text{Area of } R = \int_0^{5/2} (5-2x)(2+3x)\,dx}$$

b) $\int_0^{5/2} (5-2x)(2+3x)\,dx = \frac{225}{8}$ from GDC

$$\boxed{\text{Area of } R = \frac{225}{8} \text{ units}^2} = 28.125$$

c) Area of triangle $= \frac{1}{2} \times$ base $\times$ height

$\frac{1}{2}(9)(h) = \frac{9}{2}h = \frac{225}{8}$

$h = \frac{225}{8} \div \frac{9}{2} = \frac{225}{8} \times \frac{2}{9}$

$$\boxed{h = \frac{25}{4}} = 6.25$$

Question 7

a) P has a stationary point where $\frac{dP}{dx} = 0$

$-0.02x + 6 = 0 \Rightarrow 0.02x = 6 \Rightarrow x = 300$

Check that that is a maximum!

When $x = 299$, $-0.02(299) + 6 = 0.02 > 0$

When $x = 301$, $-0.02(301) + 6 = -0.02 < 0$

So $x = 300$ is a maximum

Because derivative goes from positive to negative

$\boxed{\text{Selling 300 kg of rice each week will maximise profit.}}$

b) $P(x) = \int \frac{dP}{dx}\, dx$

constant of integration

$P(x) = \int (-0.02x + 6)\, dx = \frac{-0.02}{2}x^2 + 6x + c$

$= -0.01x^2 + 6x + c$

But $P = 480$ when $x = 250$, so

$-0.01(250)^2 + 6(250) + c = 480$

$-0.01(62500) + 6(250) + c = 480$

$-625 + 1500 + c = 480$

$c + 875 = 480 \Rightarrow c = -395$

$$\boxed{P(x) = -0.01x^2 + 6x - 395}$$

Question 8

a) P has a stationary point where $\frac{dP}{dx} = 0$

$-1.9x + 145 = 0 \implies 1.9x = 145$

$\implies x = \frac{1450}{19} = 76.315789\ldots$

Check that that is a maximum!

When $x = 76$, $-1.9(76) + 145 = 0.6 > 0$

When $x = 77$, $-1.9(77) + 145 = -1.3 < 0$

So $x = \frac{1450}{19}$ is a maximum

Because derivative goes from positive to negative

$100\left(\frac{1450}{19}\right) = \frac{145000}{19} = 7631.578947\ldots$

Don't forget that x is <u>hundreds</u> of litres!

To the nearest litre, selling 7632 litres of paint each week will maximise profit.

b) $P(x) = \int \frac{dP}{dx}\, dx$

constant of integration

$$P(x) = \int (-1.9x + 145)\, dx = \frac{-1.9}{2}x^2 + 145x + c$$

$$= -0.95x^2 + 145x + c$$

$\frac{7000}{100} = 70$ Don't forget that x is hundreds of litres!

So $P = 5000$ when $x = 70$

$$-0.95(70)^2 + 145(70) + c = 5000$$

$$-0.95(4900) + 145(70) + c = 5000$$

$$-4655 + 10150 + c = 5000$$

$$c + 5495 = 5000 \Rightarrow c = -495$$

$$\boxed{P(x) = -0.95x^2 + 145x - 495}$$

Question 9 a) (i) Point V is on the x-axis, so at point V $y = 0$

$$9(x-6)^2 = 0 \Rightarrow (x-6)^2 = 0 \Rightarrow x = 6$$

$$\boxed{V \text{ is the point } (6, 0)}$$

(ii) $11 \times 3.6 = 39.6$

$$\boxed{\text{Area of OABC} = 39.6 \text{ m}^2}$$

b) Because OABC is a rectangle, A must be the point (0, 3.6). So when $x = 0$, $y = 3.6$.

$$q(0-6)^2 = 3.6$$

$$q(-6)^2 = 3.6$$

$$36q = 3.6$$

$$q = \frac{3.6}{36}$$

$$\boxed{q = \frac{1}{10}} = 0.1$$

c) The shaded area is the area of OABC minus $\int_0^{11} \frac{1}{10}(x-6)^2\,dx$.

$$\int_0^{11} \frac{1}{10}(x-6)^2\,dx = \frac{341}{30} \quad \text{from GDC}$$

$$39.6 - \frac{341}{30} = \frac{847}{30} = 28.233333\ldots$$

The cross-sectional area is

$$\frac{847}{30}\ \text{m}^2 = 28.2\ \text{m}^2\ (3\ \text{s.f.})$$

Question 10

a) The curve goes through (30, 15), so when $x = 30$, $y = 15$.

$$r(30-15)^2 = 15$$

$$r(15)^2 = 15$$

$$225r = 15$$

$$r = \frac{15}{225}$$

$$\boxed{r = \frac{1}{15}}$$

b) The shaded area is the area of OABC minus $\int_0^{30} \frac{1}{15}(x-15)^2 \, dx$

Area of OABC $= 30 \times 15 = 450$

$\int_0^{30} \frac{1}{15}(x-15)^2 \, dx = 150$ from GDC

$450 - 150 = 300$

The cross-sectional area is 300 cm^2

c) $1.2 \text{ m} = 120 \text{ cm}$ Don't forget to convert units!

$300 \times 120 = 36000$

Volume $= 36000$ cm^3