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5.2 Further Differentiation

IB Maths - Revision Notes

5.2.1 Differentiating Special Functions

Differentiating Trig Functions

How do I differentiate sin, cos and tan?

- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$
- The derivative of $y = \tan x$ is $\frac{dy}{dx} = \frac{1}{\cos^2 x}$
  - This result can be derived using **quotient rule**
- All three of these derivatives are given in the **formula booklet**
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = \sin(ax+b)$ is $\frac{dy}{dx} = a\cos(ax+b)$
  - the derivative of $y = \cos(ax+b)$ is $\frac{dy}{dx} = -a\sin(ax+b)$
  - **the derivative of** $y = \tan(ax+b)$ is $\frac{dy}{dx} = \frac{a}{\cos^2(ax+b)}$
- For the **general** function $f(x)$,
  - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
  - the derivative of $y = \tan(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{\cos^2(f(x))}$
- These last three results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
  - Ensure you know how to change the angle mode on your GDC

Exam Tip

- As soon as you see a question involving differentiation and trigonometry put your GDC into radians mode

Worked example

a) Find $f'(x)$ for the functions

i. $f(x) = \sin x$
ii. $f(x) = \cos(5x + 1)$

i. $\boxed{f'(x) = \cos x}$

ii. $\boxed{f'(x) = -5\sin(5x+1)}$ (Linear function $ax+b$)

b) A curve has equation $y = \tan\left(6x^2 - \frac{\pi}{4}\right)$.

Find the gradient of the tangent to the curve at the point where $x = \frac{\sqrt{\pi}}{2}$.

Give your answer as an exact value.

This is of the form $y = \tan(f(x))$

so $\frac{dy}{dx} = \frac{f'(x)}{\cos^2(f(x))}$

$f(x) = 6x^2 - \frac{\pi}{4}$

$\therefore f'(x) = 12x$

$\therefore \frac{dy}{dx} = \frac{12x}{\cos^2\left(6x^2 - \frac{\pi}{4}\right)}$

At $x = \frac{\sqrt{\pi}}{2}$, $\frac{dy}{dx} = \frac{12\left(\frac{\sqrt{\pi}}{2}\right)}{\cos^2\left[6\left(\frac{\sqrt{\pi}}{2}\right)^2 - \frac{\pi}{4}\right]}$

$= \frac{6\sqrt{\pi}}{\cos^2\left(\frac{5\pi}{4}\right)}$

$\boxed{\therefore \frac{dy}{dx} = 12\sqrt{\pi} \text{ at } x = \frac{\sqrt{\pi}}{2}}$

Differentiating e^x & lnx

How do I differentiate exponentials and logarithms?

- The derivative of $y = e^x$ is $\frac{dy}{dx} = e^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = e^{(ax+b)}$ is $\frac{dy}{dx} = ae^{(ax+b)}$
  - the derivative of $y = \ln(ax + b)$ is $\frac{dy}{dx} = \frac{a}{(ax+b)}$
    - in the special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$ (a's cancel)
- For the **general** function $f(x)$,
  - the derivative of $y = e^{f(x)}$ is $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - the derivative of $y = \ln(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
- The last two sets of results can be derived using the **chain rule**

Exam Tip

- Remember to avoid the common mistakes:
  - the derivative of $\ln kx$ with respect to x is $\frac{1}{x}$, NOT $\frac{k}{x}$
  - the derivative of e^{kx} with respect to x is ke^{kx}, NOT kxe^{kx-1}

Worked example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where $x = 2$ gving your answer in the form $y = a + be^{c}$, where a, b and c are integers to be found.

$$y = e^{-3x+1} + 2(\ln 5x)$$

$$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$$

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"

"$y = \ln(ax+b)$, special case $b=0$, $\frac{dy}{dx} = \frac{1}{x}$"

At $x = 2$, $\frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$

Your GDC may be able to find gradients but probably not in the form required. It is still helpful to check approximate answers though.

$\therefore$ Gradient at $x = 2$ is $1 - 3e^{-5}$

i.e. $a = 1$, $b = -3$, $c = -5$

5.2.2 Techniques of Differentiation

Chain Rule

What is the chain rule?

- The **chain rule** states if y is a function of u and u is a function of x then

$$y = f(u(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x))g'(x)$$

How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually x) does not 'appear alone'
    - $\sin x$ – **not** a composite function, x 'appears alone'
    - $\sin(3x+2)$ **is a composite function;** x is tripled and has 2 added to it before the sine function is applied

How do I use the chain rule?

STEP 1
Identify the two functions
Rewrite y as a function of u; $y = f(u)$
Write u as a function of x; $u = g(x)$

STEP 2

Differentiate y with respect to u to get $\frac{\mathrm{d}y}{\mathrm{d}u}$

Differentiate u with respect to x to get $\frac{\mathrm{d}u}{\mathrm{d}x}$

STEP 3

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$ and substitute u back in for $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

Are there any standard results for using chain rule?

- There are **five** general results that can be useful
    - If $y = (f(x))^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = nf'(x)\mathrm{f}(x)^{n-1}$
    - If $y = \mathrm{e}^{f(x)}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)e^{f(x)}$
    - If $y = \ln(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)}{f(x)}$
    - If $y = \sin(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)\cos(f(x))$
    - If $y = \cos(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -f'(x)\sin(f(x))$

Exam Tip

- You should aim to be able to spot and carry out the chain rule mentally (rather than use substitution)
    - every time you use it, say it to yourself in your head
      "differentiate the first function ignoring the second, then multiply by the derivative of the second function"

Worked example

a) Find the derivative of $y=(x^2-5x+7)^7$.

STEP 1 Identify the two functions and rewrite

$y = u^7$ i.e. $f(u)=u^7$

$u = x^2-5x+7$ i.e. $g(x)=x^2-5x+7$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

$\frac{dy}{du} = 7u^6$ $\frac{du}{dx} = 2x-5$

STEP 3 Apply chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$\frac{dy}{dx} = 7u^6(2x-5)$

and substitute u back for $g(x)$

$$\boxed{\frac{dy}{dx} = 7(2x-5)(x^2-5x+7)^6}$$

b) Find the derivative of $y=\sin(e^{2x})$.

$y = \sin(e^{2x})$ "... differentiate $\sin\square$, ignore e^{2x}..."

$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$ "... multiply by derivative of e^{2x}..."

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"
or by applying chain rule again

$$\boxed{\therefore \frac{dy}{dx} = 2e^{2x}\cos(e^{2x})}$$

Product Rule

What is the product rule?

- The **product rule** states if y is the product of two functions $u(x)$ and $v(x)$ then

$$y = uv$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g'(x) + g(x)f'(x)$$

- **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of x"
    - $\sin x \cos x$ is a product, "sin x times cos x"

How do I use the product rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the product rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v'

Exam Tip

- Use u, v, u' and v' for the elements of product rule
  - lay them out in a 'square' (imagine a 2x2 grid)
  - those that are paired together are then on opposite diagonals (u and v', v and u')
- For trickier functions chain rule may be reuqired inside product rule
  - i.e. chain rule may be needed to differentiate u and v

Worked example

a) Find the derivative of $y = e^x \sin x$.

$y = e^x \sin x$

STEP 1 Identify functions and differentiate

$u = e^x \quad v = \sin x$

$u' = e^x \quad v' = \cos x$

arranging u, v, u', v' in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule: '$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$'

(As it is given in the formula booklet)

$y' = e^x \cos x + e^x \sin x$

$\boxed{\therefore \frac{dy}{dx} = e^x(\cos x + \sin x)}$

It is straightforward to take a factor of e^x out

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$y = 5x^2 \cos 3x^2$

STEP 1 $\quad u = 5x^2 \quad v = \cos 3x^2$

$u' = 10x \quad v' = -\sin 3x^2 \times 6x$ chain rule

$v' = -6x \sin 3x^2$

STEP 2 $\quad y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$\boxed{\therefore \frac{dy}{dx} = 10x(\cos 3x^2 - 3x^2 \sin 3x^2)}$

Quotient Rule

What is the quotient rule?

- The **quotient rule** states if y is the quotient $\frac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of x
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

How do I use the quotient rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{dy}{dx}$ by applying the quotient rule formula $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important

Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v',

Exam Tip

- Use u, v, u' and v' for the elements of quotient rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals (v and u', u and v')
- Look out for functions of the form $y = f(x)(g(x))^{-1}$
  - These can be differentiated using a combination of **chain rule** and **product rule** (it would be good practice to try!)
  - ... but it can also be seen as a quotient rule question in disguise
  - ... and vice versa!
    - A quotient could be seen as a product by rewriting the denominator as $(g(x))^{-1}$

Worked example

Differentiate $f(x)=\dfrac{\cos 2x}{3x+2}$ with respect to x.

STEP 1 Identify u and v, differentiate

$u=\cos 2x \qquad v=3x+2$

$u'=-2\sin 2x \qquad v'=3$

chain rule

opposite diagonals match up

STEP 2 Apply quotient rule: '$\dfrac{dy}{dx}=\dfrac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$'

(As it is given in the formula booklet)

$$f'(x)=\frac{(3x+2)(-2\sin 2x)-(\cos 2x)(3)}{(3x+2)^2}$$

$$\boxed{\therefore\ f'(x)=\frac{-2(3x+2)\sin 2x-3\cos 2x}{(3x+2)^2}}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)

5.2.3 Related Rates of Change

Related Rates of Change

What is meant by rates of change?

- A **rate of change** is a measure of how a quantity is changing with respect to another quantity
- Mathematically rates of change are **derivatives**
  - $\frac{\mathrm{d}V}{\mathrm{d}r}$ could be the rate at which the volume of a sphere changes relative to how its radius is changing
- Context is important when interpreting positive and negative rates of change
  - A positive rate of change would indicate an increase
    - e.g. the change in volume of water as a bathtub fills
  - A negative rate of change would indicate a decrease
    - e.g. the change in volume of water in a leaking bucket

What is meant by related rates of change?

- **Related rates of change** are connected by a linking variable or parameter
  - this is usually **time**, represented by t
  - **seconds** is the standard unit for time but this will depend on context
- e.g. Water running into a large bowl
  - both the height and volume of water in the bowl change with time
  - time is the linking parameter

How do I solve problems involving related rates of change?

- Use of chain rule

$$y = g(u) \quad u = f(x) \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

- Chain rule is given in the **formula booklet** in the format above
  - **Different letters** may be used relative to the context
    - e.g. V for **volume**, S for **surface area**, h for **height**, r for **radius**
- Problems often involve one quantity being **constant**
  - so another quantity can be expressed in terms of a **single** variable
  - this makes finding a derivative a lot easier
- For **time** problems at least, it is more convenient to use

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}y}{\mathrm{d}x}$$

and if it is more convenient to find $\frac{\mathrm{d}x}{\mathrm{d}y}$ than $\frac{\mathrm{d}y}{\mathrm{d}x}$ then use chain rule in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}y}$$

- **Neither** of these alternative versions of chain rule are in the **formula booklet**

  STEP 1
  Write down the rate of change given and the rate of change required
  (If unsure of the rates of change involved, use the units given as a clue
  e.g. $\mathrm{m\,s^{-1}}$ (metres per second) would be the rate of change of length, per time, $\frac{\mathrm{d}l}{\mathrm{d}t}$)

  STEP 2
  Use chain rule to form an equation connecting these rates of change with a third rate
  The third rate of change will come from a related quantity such as volume, surface area, perimeter

  STEP 3
  Write down the formula for the related quantity (volume, etc) accounting for any fixed quantities
  Find the third rate of change of the related quantity (derivative) using differentiation

  STEP 4
  Substitute the derivative and known rate of change into the equation and solve it

Exam Tip

- If you struggle to determine which rate to use in an exam then you can look at the units to help
  - e.g. A rate of 5 **cm³** *per* **second** implies **volume** *per* **time** so the rate would be $\frac{\mathrm{d}V}{\mathrm{d}t}$

Worked example

A cuboid has a square cross-sectional area of side length x cm and a fixed height of 5 cm.
The volume of the cuboid is increasing at a rate of 20 cm^3 s^{-1}.
Find the rate at which the side length is increasing at the point when its side length is 3 cm.

STEP 1: Write down rates of change given and required

$$\frac{dV}{dt} = 20$$ (Units are cm^3 (volume) s^{-1} (per second))

$\frac{dx}{dt}$ is required

STEP 2: Form equation from chain rule and a third 'connecting' rate

$$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$$

STEP 3: Formula for linking quantity, and its derivative

Volume (of a cuboid) is the link

$V = x^2 \times 5 = 5x^2$ (Cross-section is square, height is constant)

Differentiate, $\frac{dV}{dx} = 10x$

STEP 4: Substitute and solve

$$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$$

$$20 = \frac{dx}{dt} \times 10(3)$$

x, side length is 3

$$\boxed{\therefore \frac{dx}{dt} = \frac{2}{3} \text{ cm s}^{-1}}$$

5.2.4 Second Order Derivatives

Second Order Derivatives

What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = f(x)$
  - $\frac{dy}{dx} = f'(x)$ (First order derivative)
  - $\frac{d^2y}{dx^2} = f''(x)$ (Second order derivative)
- Note the position of the superscript 2's
  - **d**ifferentiating twice (so $\mathbf{d}^2$) with respect to x twice (so x^2)
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate** of **change** of a function
  - a **second order derivative** is the **rate** of **change** of the **rate** of **change** of a function
    - i.e. the **rate** of **change** of the function's **gradient**
- **Second order derivatives** can be used to
    - test for local minimum and maximum points
    - help determine the nature of stationary points
    - determine the concavity of a function
    - graph derivatives

How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions, roots,** etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule

Exam Tip

- Negative and/or fractional powers can cause problems when finding second derivatives so work carefully through each term

Worked example

Given that $f(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a) Find $f'(x)$ and $f''(x)$.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ REWRITE AS POWERS OF x

$f'(x) = -\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} + 3\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$ DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} - \frac{3}{2}\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$ DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b) Evaluate $f''(3)$.

Give your answer in the form $a\sqrt{b}$, where b is an integer and a is a rational number.

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$ $x^{\frac{3}{2}} = x\sqrt{x}$ $x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$ RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$

5.2.5 Further Applications of Differentiation

Stationary Points & Turning Points

What is the difference between a stationary point and a turning point?

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** is a type of stationary point, but in addition the **function changes** from **increasing** to **decreasing,** or **vice versa**
  - The curve **'turns'** from **'going upwards'** to **'going downwards'** or **vice versa**
  - **Turning points** will either be **(local) minimum** or **maximum** points
- A **point** of **inflection** ***could*** also be a **stationary point** but is **not** a turning point

How do I find stationary points and turning points?

- For the function $y = f(x)$, **stationary points** can be found using the following process

  STEP 1

  Find the **gradient function,** $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$

  STEP 2

  Solve the equation $f'(x) = 0$ to find the x-coordiante(s) of any stationary points

  STEP 3

  If the y-coordaintes of the stationary points are also required then substitute the x-coordinate(s) into $f(x)$

- A GDC will solve $f'(x) = 0$ and most will find the coordinates of turning points (minimum and maximum points) in graphing mode

Testing for Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
  - i.e. $f'(x) = 0$
- A **local minimum** point, $(x,\ f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x,\ f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend** to **infinity** for **large** values of x (and/or **minus infinity** for **large negative** values of x)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of $f(x)$ for **all values** of x
  - similar for a **global** maximum point

How do I find local minimum & maximum points?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function $f(x)$ ...

  STEP 1

  Find $f'(x)$ and solve $f'(x) = 0$ to find the x-coordinates of any stationary points

  STEP 2 (Second derivative)

  Find $f''(x)$ and evaluate it at each of the stationary points found in **STEP 1**

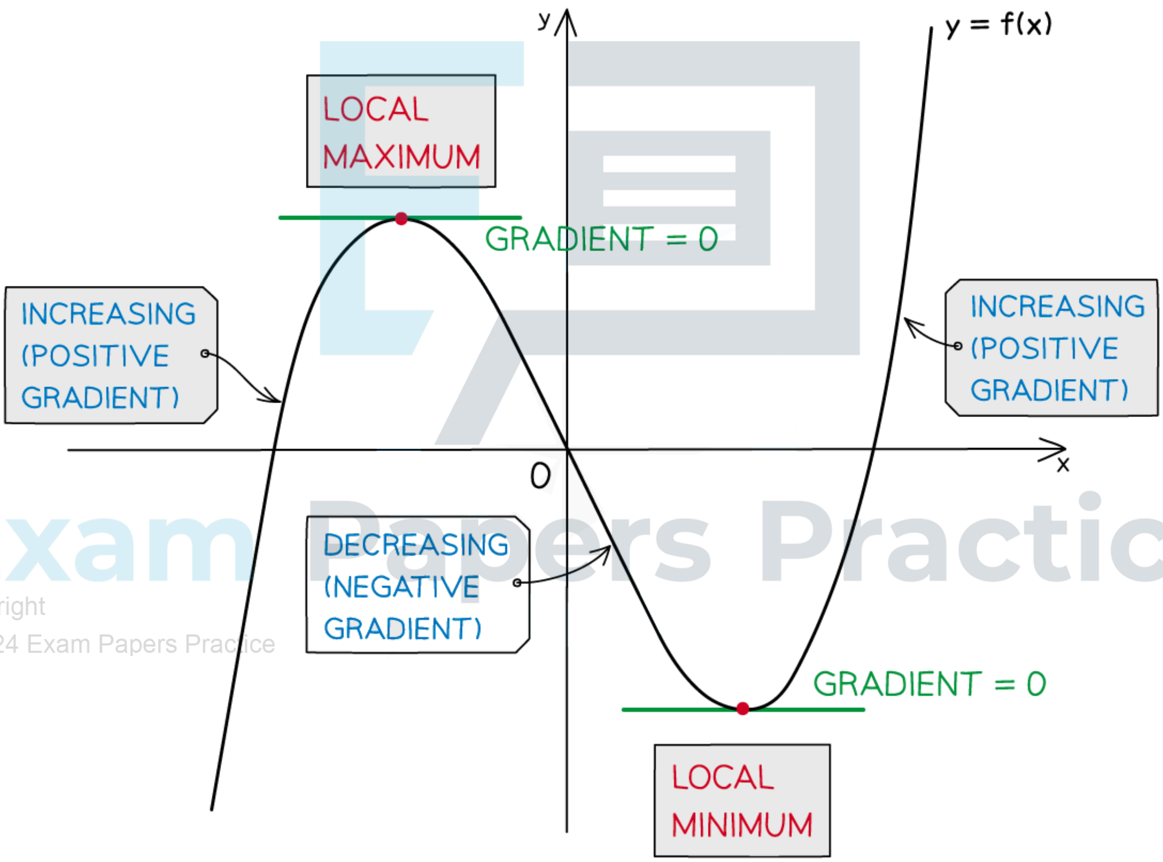
  STEP 3 (Second derivative)

  - If $f''(x) = 0$ then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
  - If $f''(x) > 0$ then the curve of the graph of $y = f(x)$ is **concave up** and the stationary point is a **local minimum** point
  - If $f''(x) < 0$ then the curve of the graph of $y = f(x)$ is **concave down** and the stationary point is a **local maximum** point

  STEP 4 (First derivative)

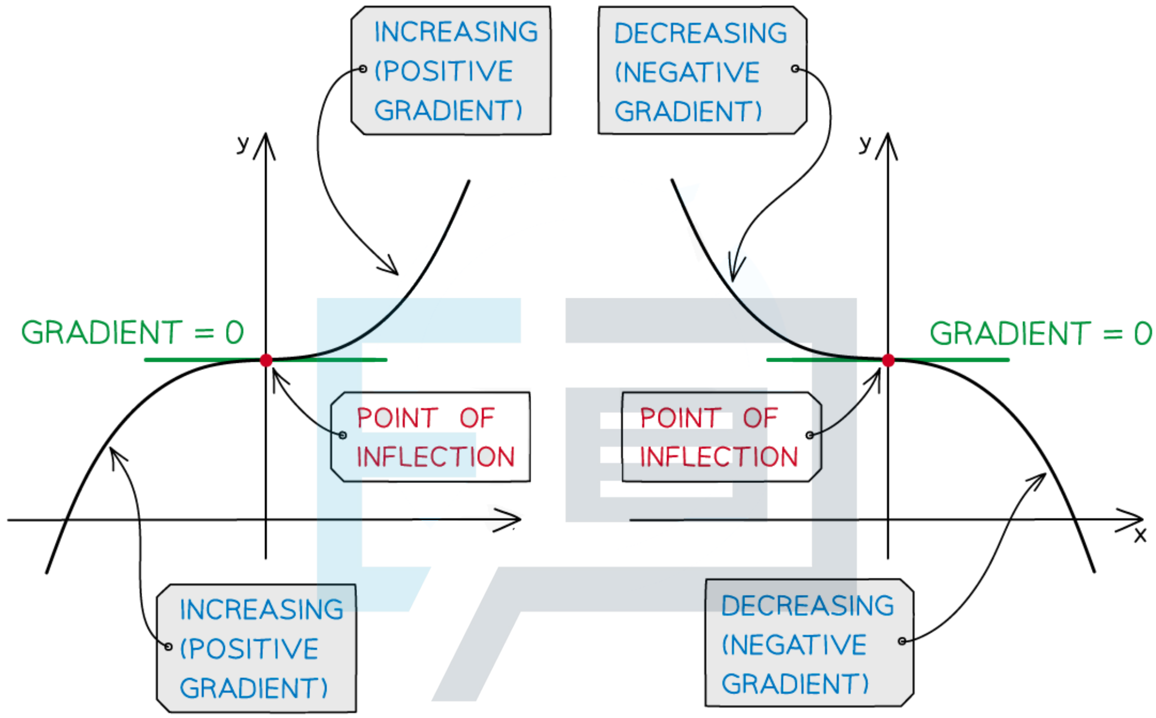
Find the sign of the first derivative just either side of the stationary point; i.e. evaluate $f'(x-h)$ and $f'(x+h)$ for small h

- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h)<0,\ f'(x)=0,\ f'(x+h)>0$
- A **local maximum point** changes the **function** from **increasing** to **decreasing**
  - the **gradient** changes from **positive** to **negative**
  - $f'(x-h)>0,\ f'(x)=0,\ f'(x+h)<0$



- A **stationary point** of **inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not change** sign
  - $f'(x-h)>0,\ f'(x+h)>0$ or $f'(x-h)<0,\ f'(x+h)<0$

- a **point** of **inflection** does **not** necessarily have $f'(x) = 0$
  - this method will only find those that do - and are often called **horizontal points** of **inflection**



Exam Tip

- Exam questions may use the phrase "classify turning points" instead of "find the nature of turning points"
- Using your GDC to sketch the curve is a valid test for the nature of a stationary point in an exam unless the question says "show that..." or asks for an algebraic method
- Even if required to show a full algebraic solution you can still use your GDC to tell you what you're aiming for and to check your work

Worked example

Find the coordinates and the nature of any stationary points on the graph of $y = f(x)$ where $f(x) = 2x^3 - 3x^2 - 36x + 25$.

At stationary points, $f'(x) = 0$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$6(x^2 - x - 6) = 0$

$(x-3)(x+2) = 0$

$x = 3, \quad y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$

$x = -2, \quad y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$

Using the second derivative to determine their nature

$f''(x) = 12x - 6 = 6(2x - 1)$

$f''(3) = 6(2 \times 3 - 1) = 30 > 0$

$\therefore x = 3$ is a local minimum point

$f''(-2) = 6(2 \times -2 - 1) = -30 < 0$

$\therefore x = -2$ is a local maximum point

(Note: In this case, both stationary points are turning points)

Turning points are:

$(3, -56)$ local minimum point

$(-2, 69)$ local maximum point

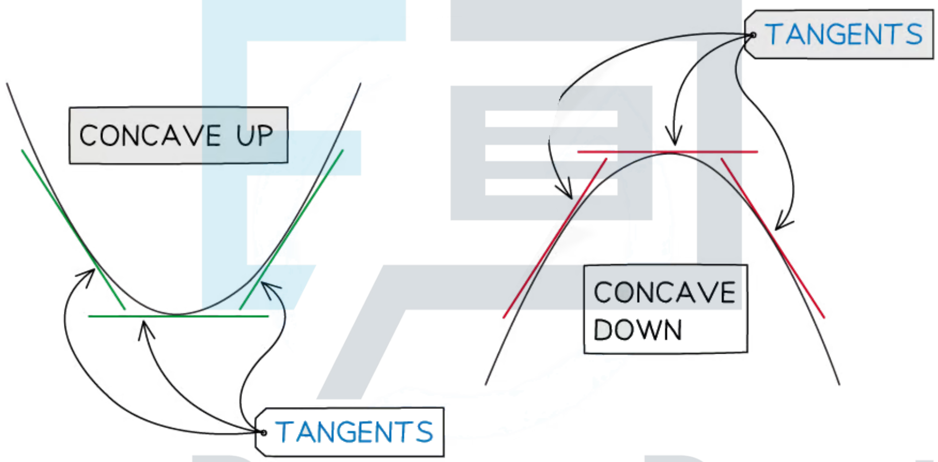
Use a GDC to graph $y = f(x)$ and the max/min solving feature to check the answers.

5.2.6 Concavity & Points of Inflection

Concavity of a Function

What is concavity?

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if $f''(x) < 0$ for all values of x in an interval
  - a curve is **CONCAVE UP** if $f''(x) > 0$ for all values of x in an interval



Exam Tip

- In an exam an easy way to remember the difference is:
  - Concave **down** is the shape of (the mouth of) a sad smiley ☹
  - **Concave up** is the shape of (the mouth of) a happy smiley ☺

Worked example

The function $f(x)$ is given by $f(x) = x^3 - 3x + 2$.

a) Determine whether the curve of the graph of $y = f(x)$ is concave down or concave up at the points where $x = -2$ and $x = 2$.

$f(x) = x^3 - 3x + 2$
$f'(x) = 3x^2 - 3$
$f''(x) = 6x$

$f''(-2) = 6 \times -2 = -12 < 0$ (concave down)

$f''(2) = 6 \times 2 = 12 > 0$ (concave up)

$$\boxed{\begin{array}{l}\text{At } x = -2, \quad y = f(x) \text{ is concave down} \\ \text{At } x = 2, \quad y = f(x) \text{ is concave up}\end{array}}$$

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answers are sensible

b) Find the values of x for which the curve of the graph $y = f(x)$ of is concave up.

$f''(x) = 6x$ from part (a)

Concave up is $f''(x) > 0$

$6x > 0$ when $x > 0$

$$\boxed{\therefore\ y = f(x) \text{ is concave up for } x > 0}$$

Use your GDC to check your answer

Points of Inflection

What is a point of inflection?

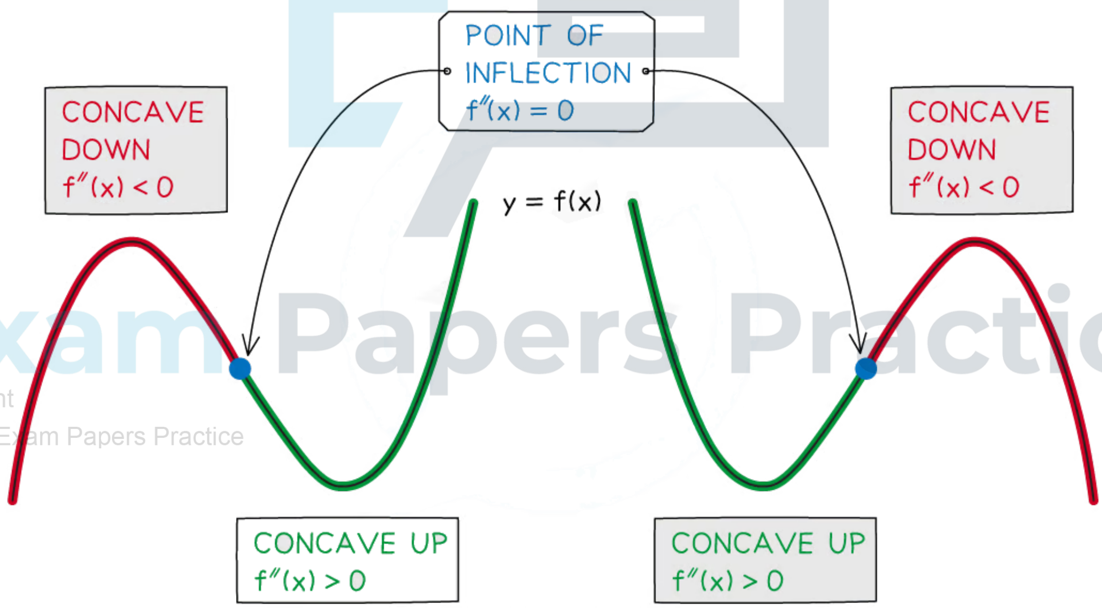
- A point at which the curve of the graph of $y = f(x)$ changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion,** may sometimes be used

What are the conditions for a point of inflection?

- A point of inflection requires **BOTH** of the following two conditions to hold
  - the **second derivative** is zero
    - $f''(x) = 0$

  AND

  - the graph of $y = f(x)$ changes **concavity**
    - $f''(x)$ changes **sign** through a **point** of **inflection**



- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
  - points where $f''(x) = 0$ could be **local minimum** or **maximum** points
    - the **first derivative** test would be needed
  - However, if it is already known $f(x)$ has a point of inflection at $x = a$, say, then $f''(a) = 0$

What about the first derivative, like with turning points?

- A **point** of **inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ($f'(x) = 0$)
  - If it does, it is also a **stationary point** and is often called a **horizontal point** of **inflection**
    - the tangent to the curve at this point would be horizontal
  - The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

How do I find the coordinates of a point of inflection?

- For the function $f(x)$

  STEP 1

  Differentiate $f(x)$ **twice** to find $f''(x)$ and solve $f''(x) = 0$ to find the x-coordinates of possible points of inflection

  STEP 2

  Use the **second derivative** to **test** the **concavity** of $f(x)$ either side of $x = a$

  - If $f''(x) < 0$ then $f(x)$ is concave down
  - If $f''(x) > 0$ then $f(x)$ is concave up

  If concavity changes, $x = a$ is a **point of inflection**

  STEP 3

  If required, the y-coordinate of a point of inflection can be found by substituting the x-coordinate into $f(x)$

Exam Tip

- You can find the *x*-coordinates of the point of inflections of $y = f(x)$ by drawing the graph $y = f'(x)$ and finding the *x*-coordinates of any local maximum or local minimum points
- Another way is to draw the graph $y = f''(x)$ and find the *x*-coordinates of the points where the graph crosses (not just touches) the *x*-axis

Worked example

Find the coordinates of the point of inflection on the graph of $y = 2x^3 - 18x^2 + 24x + 5$. Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve $f''(x)=0$

$f(x) = 2x^3 - 18x^2 + 24x + 5$

$f'(x) = 6x^2 - 36x + 24$

$f''(x) = 12x - 36$

$12x - 36 = 0$ when $x = 3$

STEP 2: Use the second derivative to test concavity

$f''(3) = 0$

$f''(2.9) < 0$ (concave down)

$f''(3.1) > 0$ (concave up)

$\therefore$ concavity changes through $x = 3$

STEP 3: The y-coordinate is required

$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

Since $f''(3)=0$ AND the graph of $y=f(x)$ changes concavity through $x=3$, the point $(3,-31)$ is a point of inflection.

Use your GDC to plot the graph of $y=f(x)$ and to help see if your answer is sensible