Exam Papers Practice

5.1 Differentiation

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Course	DP IB Maths
Section	5. Calculus
Topic	5.1 Differentiation
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1 a) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$y = \frac{3}{2}x^2 - 15x + 2$

Apply formula

$$\boxed{\frac{dy}{dx} = 3x - 15}$$

b)i) Set $\frac{dy}{dx} = -3$ and solve for x.

$3x - 15 = -3$ add 15 then divide by 3

$x = 4$

Sub $x = 4$ into y.

$y = \frac{3}{2}(4)^2 - 15(4) + 2$

$y = -34$

$$\boxed{\therefore A(4, -34)}$$

ii) Sub A and $m = -3$ into $y - y_1 = m(x - x_1)$.

$y - (-34) = -3(x - 4)$ expand and rearrange

$$\boxed{y = -3x - 22}$$

Question 2

a) Derivative of x^n formula (in formula booklet)

$$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$$

$$f(x) = 3x^7 - 12x$$

Apply formula

$$\boxed{f'(x) = 21x^6 - 12}$$

b) Sub $x = 0$ into $f'(x)$.

$$f'(0) = 21(0)^6 - 12$$

$$\boxed{f'(0) = -12}$$

c) The tangent and normal are perpendicular.

if $m_{normal} = 4$ then $m_{tangent} = -\frac{1}{4}$

Set $f'(x) = -\frac{1}{4}$ and solve for x.

$$21x^6 - 12 = -\frac{1}{4}$$

add 12, divide by 21 then $\sqrt[6]{\ }$.

$$x = \pm\sqrt[6]{\frac{11.75}{21}} = \pm 0.9072\ldots$$

$$x = \pm 0.908 \quad (3sf)$$

$$\therefore y = \pm 9.3694\ldots = \pm 9.37 \quad (3sf)$$

$$\boxed{(0.908, -9.37) \text{ and } (-0.908, 9.37)}$$

Question 3

a) Find $\frac{dy}{dx}$

$y = 4 - \frac{4}{x} = 4 - 4x^{-1}$

$\frac{dy}{dx} = \frac{4}{x^2} = 4x^{-2}$

Sub $x = 2$ into $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{4}{(2)^2} = \frac{4}{4} = 1 \qquad \therefore m = 1$

Sub $x = 2$ into y.

$y = 4 - \frac{4}{(2)} = 2 \qquad \therefore$ point $(2, 2)$

Sub m and the point into $y - y_1 = m(x - x_1)$.

$y - 2 = 1(x - 2)$

$\boxed{y = x}$

b) Set $\frac{dy}{dx} = 16$ and solve for x.

$\frac{4}{x^2} = 16$

reciprocate and multiply by 4

$x^2 = \frac{1}{4}$

$\sqrt{\ }$

$x = \pm\frac{1}{2}$

$\therefore y = -4$ and 12.

$\boxed{\left(\tfrac{1}{2}, -4\right) \text{ and } \left(-\tfrac{1}{2}, 12\right)}$

Question 4

a)i) Sub $x = 2$ into $f(x)$.

$$f(2) = \frac{4}{(2)} + \frac{2(2)^4}{5} - \frac{2}{5}$$

$$\boxed{f(2) = 8}$$

ii) Find $f'(x)$

$$f(x) = 4x^{-1} + \frac{2}{5}x^4 - \frac{2}{5}$$

$$f'(x) = -4x^{-2} + \frac{8}{5}x^3$$

Sub $x = 2$ into $f'(x)$.

$$f'(2) = -4(2)^{-2} + \frac{8}{5}(2)^3$$

$$\boxed{f'(2) = 11.8}$$

b) point $(2, 8)$ $m = 11.8$

Sub m and the point into $y - y_1 = m(x - x_1)$.

$$y - 8 = 11.8(x - 2)$$

expand and rearrange

$$y = 11.8x - 15.6$$

$$\boxed{\text{The equation of } l \text{ is } y = 11.8x - 15.6.}$$

c) Graph $f(x)$ and l on your GDC and find their intersection.

$\boxed{A(-0.222, -18.2)}$

Question 5 a) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$f(x) = x^2 - bx + c$

Apply formula

$\boxed{f'(x) = 2x - b}$

b) Tangent equation at $x = 2$ is $y = x - 1$.

$\therefore f'(2) = 1$

$2(2) - b = 1$

$\boxed{b = 3}$

c) Sub $x = 2$ into $y = x - 1$.

$y = 2 - 1$

$y = 1$

$\therefore f(x)$ passes through $(2, 1)$.

$f(2) = 1$

$(2)^2 - 3(2) + c = 1$

$\boxed{\begin{array}{l} c = 3 \\ f(x) = x^2 - 3x + 3 \end{array}}$

Question 6

a) $\frac{dy}{dx} = 2ax^{2-1} + bx^{1-1} + 0$

$= 2ax + b$

Sub in each gradient and its corresponding x value

$-7 = 2a(-1) + b$

$\boxed{-7 = -2a + b}$

$-3 = 2a(1) + b$

$\boxed{-3 = 2a + b}$

b) Find 1 + 2 to eliminate a

$2a + b - 2a + b = -3 - 7$

$2b = -10$

$\boxed{b = -5}$

Sub $b = -5$ into 1

$2a - 5 = -3$

$2a = 2$

$\boxed{a = 1}$

c) Sub in values of x, y, a and b into eqn to find c

$$(13) = (1)(-1)^2 + (-5)(-1) + c$$

$$13 = 1 + 5 + c$$

$$\boxed{c = 7}$$

Question 7

a) $y = 3x^2 - 6 + 4x^{-1}$

$$\boxed{\frac{dy}{dx} = 6x - 4x^{-2}}$$

b) Find gradient at P by subbing $x = 2$ into $\frac{dy}{dx}$ eqn.

$$\boxed{y - y_1 = \frac{-1}{m}(x - x_1)}$$

$$m = \frac{dy}{dx} = 6(1) - 4(1)^{-2} = 2$$

$$y - 1 = \frac{-1}{2}(x - 1)$$

$$2y - 2 = -x + 1$$

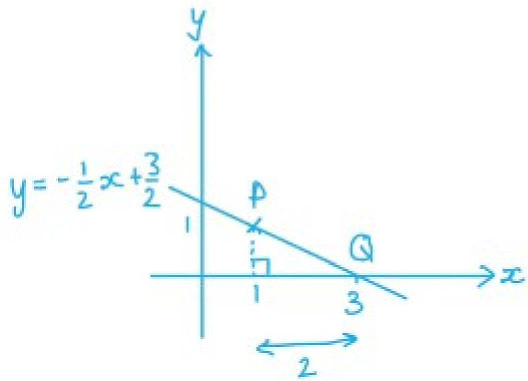
$$\boxed{x + 2y = 3}$$

c)

At Q, $y=0$

$x + 2(0) = 3$

$x = 3$



Length $PQ = \sqrt{1^2 + 2^2}$

$= \boxed{\sqrt{5}}$

Question 8

$f'(x) \geqslant 0$ for the function to be increasing

$f'(x) = -9(2)x^1 + 5 \geqslant 0$

$-18x + 5 \geqslant 0$

$\boxed{x \leqslant \frac{5}{18}}$

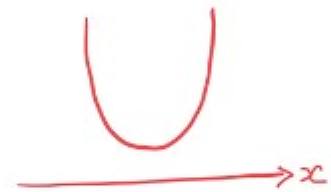
Question 9

$f'(x) \geq 0$

$f'(x) = \underline{3x^2 - 6x + 6} \geqslant 0$

positive quadratic with no real roots

$\boxed{\therefore f'(x) \text{ is always } > 0}$



This curve never crosses the x axis, so $f'(x) > 0$

OR

...An alternative way to deal with the quadratic:

1) Add 3 to both sides of inequality

$3x^2 - 6x + 9 \geqslant 3$

2) Complete the square

$3(\underbrace{(x-1)^2}_{\geqslant 0 \text{ for all } x \in \mathbb{R}} + 1) \geqslant 3$

Question 10

a) Horizontal tangents are at the local minimum and maximum (points A and B), where the gradient is 0.

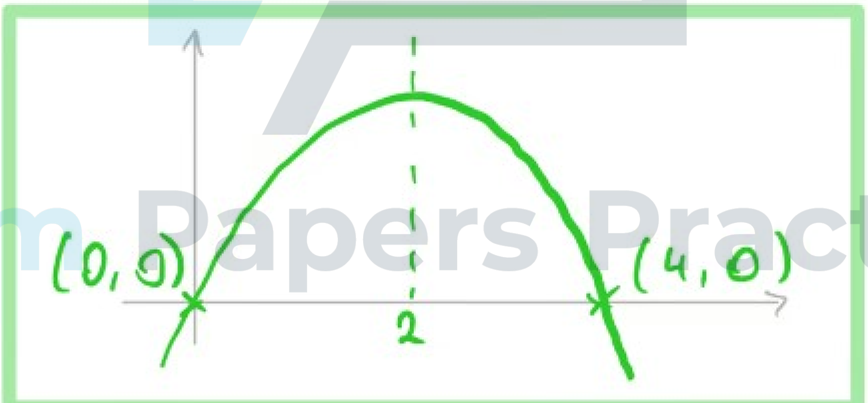
$\boxed{y = 0 \quad \text{and} \quad y = 8}$

b) Point of inflection is half way between the local min and max.

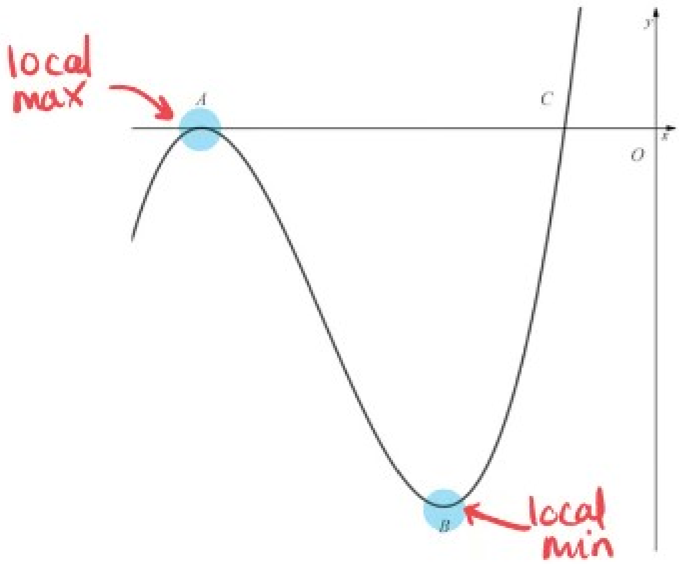
$\boxed{\therefore x = 2}$

c) f is decreasing in the intervals $x < 0$ and $x > 4$.

d) The graph of $f'(x)$ has zeros at $(0, 0)$ and $(4, 0)$ and a max when $x = 2$.



Question 11



a) 1. Find x coordinates of stationary points.

$$\frac{dy}{dx} = 0 = 3x^2 + 22x + 35$$
$$(3x+7)(x+5)$$
$$x = -\frac{7}{3}, -5$$

2. Classify the stationary points.

$$\frac{d^2y}{dx^2} = 6x + 22$$

At $x = -\frac{7}{3}$ $\quad \frac{d^2y}{dx^2} = 6\left(-\frac{7}{3}\right) + 22 = 8 > 0 \therefore$ local min $\rightarrow$ B

At $x = -5$ $\quad \frac{d^2y}{dx^2} = 6(-5) + 22 = -8 < 0 \therefore$ local max $\rightarrow$ A

3. Find corresponding y values.

At $x = -\frac{7}{3}$ $\quad y = \left(-\frac{7}{3}\right)^3 + 11\left(-\frac{7}{3}\right)^2 + 35\left(-\frac{7}{3}\right) + 25 = \frac{-256}{27}$

At $x = -5$ $\quad y = (-5)^3 + 11(-5)^2 + 35(-5) + 25 = 0$

$$\boxed{A\,(-5, 0) \qquad B\left(-\frac{7}{3}, \frac{-256}{27}\right)}$$

b) When $x = -1$,

$$y = (-1)^3 + 11(-1)^2 + 35(-1) + 25$$

$$= -1 + 11 - 35 + 25$$

$$= 0$$

$\therefore$ $(-1, 0)$ Lies on the curve

This has to be C, because the only other point where $y = 0$ is $A(-5, 0)$, and the curve is a cubic, $\therefore$ it's not going to double back and cross the x axis again!

Question 12 a) Derivative of x^n formula (in formula booklet)

$$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$$

$$y = \frac{1}{35}x^5 - \frac{3}{4}x^3 + 6x$$

Apply formula

$$\boxed{\frac{dy}{dx} = \frac{1}{7}x^4 - \frac{9}{4}x^2 + 6}$$

b) The tangent and normal line are perpendicular.

$m_{normal} = -\frac{1}{10}$ $\therefore m_{tangent} = 10$

Set $\frac{dy}{dx} = 10$ and solve for x.

$\frac{1}{7}x^4 - \frac{9}{4}x^2 + 6 = 10$

$x = \pm 4.1668... = \pm 4.17$ (3sf)

x-coordinate for R is 4.17 and the x-coordinate for S is -4.17.

Question 13

a) Stationary points occur where the gradient, $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\boxed{x = 3, 1}$

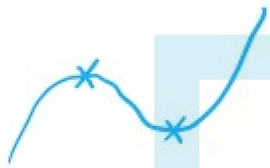
b) $\frac{d^2y}{dx^2} = 6x - 12$

When $x = 1$

$\frac{d^2y}{dx^2} = 6(1) - 12 = -6 < 0 \therefore$ maximum point

When $x = 3$

$\frac{d^2y}{dx^2} = 6(3) - 12 = 6 > 0 \therefore$ minimum point



This can even be predicted by considering where the stationary points are on a positive cubic curve!

There is a maximum point at $x = 1$ and a minimum point at $x = 3$.

c) For a point of inflection, $\frac{d^2y}{dx^2} = 0$,

and $\frac{d^2y}{dx^2}$ changes sign either side.

$\frac{dy}{dx} = 3x^2 - 12x + 9$

$\frac{d^2y}{dx^2} = 6x - 12 = 0$

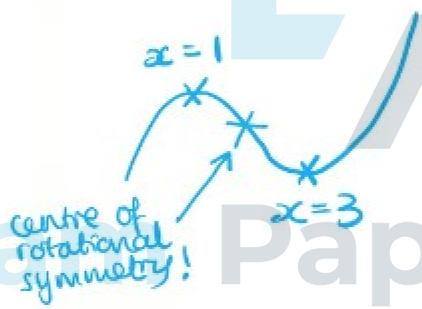
$x = 2$

When $x = 0$ $\quad \frac{d^2y}{dx^2} = 6(0) - 12 = -12 < 0$

When $x = 3$ $\quad \frac{d^2y}{dx^2} = 6(3) - 12 = 6 > 0$

$\boxed{\text{Change of sign indicates } x = 2 \text{ is a POI.}}$

OR



Since this is a +ve cubic, the POI lies midway between the stationary points $\therefore x = 2$.

d) Several possible answers...

- It was not a solution in part (a)
- when $x = 2$, $\frac{dy}{dx} \neq 0$

Question 14

The graph of a continuous function has the following properties:

The function is concave in the interval $(-\infty, a)$.

The function is convex in the interval (a, ∞).

The graph of the function intercepts the x-axis at the points $(b, 0)$, $(c, 0)$ and $(d, 0)$, where b, c and d are such that $d > c > b > 0$.

The x-coordinates of the turning points of the function are e and f, which are such that $f > e$.

The graph of the function intercepts the y-axis at $(0, g)$

Given that the value of the function is positive when $x = a$, sketch a graph of the function. Be sure to label the x-axis with the x-coordinates of the stationary points and the point of inflection, and also to label the points where the graph crosses the coordinate axes.

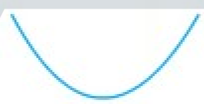
[4]

A stationary point is a point where a function's gradient is zero. This includes (but is not limited to) turning points, i.e. local maximums and minimums.

A point of inflection is where a function changes from concave to convex or vice versa.

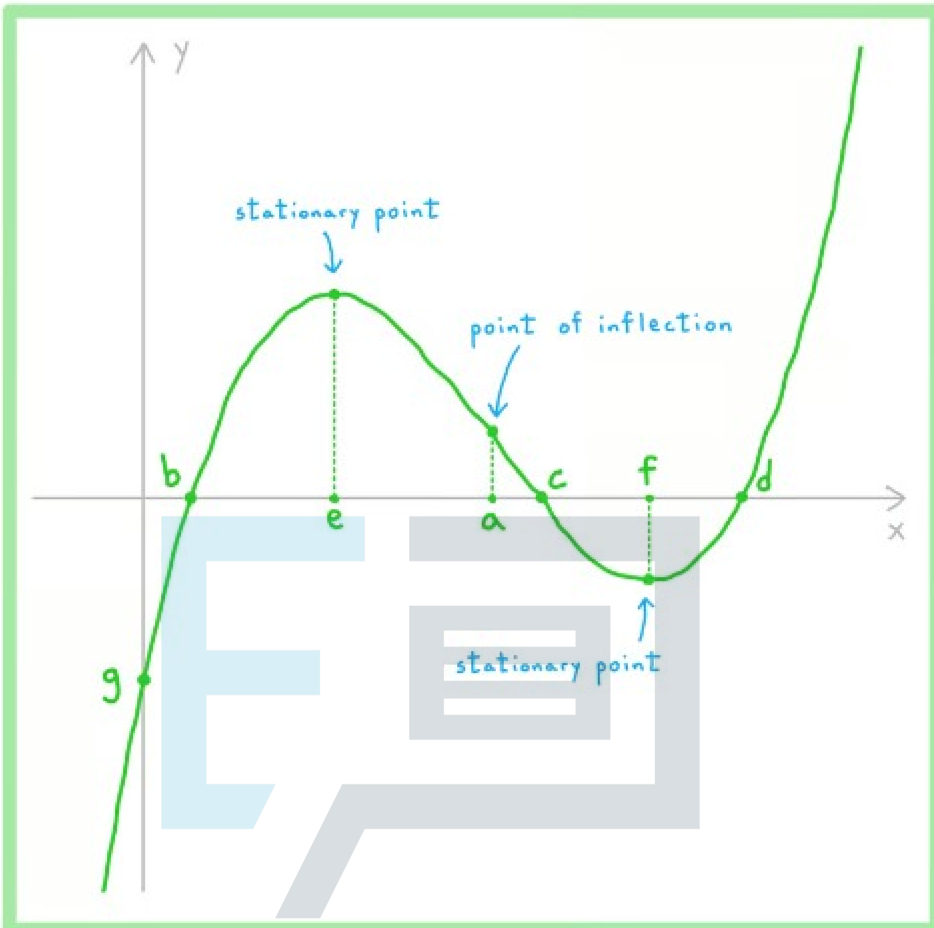
concave

(sometimes called 'concave down')



convex

(sometimes called 'concave up')


y
stationary point
point of inflection
b
c
f
d
e
a
x
stationary point
g