# DP IB Maths: AA HL

## 5.12 Further Limits (inc l'Hôpital's Rule)

### Contents

## 5.12.1 Further Limits

# l'Hôpital's Rule

### What is l'Hôpital's Rule?

- **l'Hôpital's rule** is a method involving calculus that allows us to find the value of certain limits
- Specifically, it allows us to attempt to evaluate the limit of a quotient $\frac{f(x)}{g(x)}$ for which our usual limit evaluation techniques would return one of the **indeterminate forms** $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

### How do I evaluate a limit using l'Hôpital's Rule?

- STEP 1: Check that the limit of the quotient results in one of the indeterminate forms given above
  - I.e., check that $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
- STEP 2: Find the derivatives of the numerator and denominator of the quotient
- STEP 3: Check whether the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists
- STEP 4: If that limit does exist, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
- STEP 5: If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then you may repeat the process by considering $\lim_{x \to a} \frac{f''(x)}{g''(x)}$ (and possibly higher order derivatives after that)
  - As long as the limits continue giving indeterminate forms you may continue applying l'Hôpital's rule
  - Each time this happens find the next set of derivatives and consider the limit again

## Worked example

Use l'Hôpital's rule to evaluate each of the following limits:

a) $\lim_{x \to 0} \frac{\sin x}{e^x - 1}$.

STEP 1: $\lim_{x \to 0} \frac{\overbrace{\sin x}^{f(x)}}{\underbrace{e^x - 1}_{g(x)}} = \frac{\sin(0)}{e^0 - 1} = \frac{0}{0}$ ← indeterminate form

STEP 2: $\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(e^x - 1) = e^x$

STEP 3: $\lim_{x \to 0} \frac{\cos x}{e^x} = \frac{\cos(0)}{e^0} = \frac{1}{1} = 1$ ← limit exists

STEP 4: $\boxed{\lim_{x \to 0} \frac{\sin x}{e^x - 1} = 1}$

b) $\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x}$.

STEP 1: $\lim_{x \to 0} \dfrac{x^3}{-2x + \sin 2x} = \dfrac{0^3}{-2(0) + \sin 0} = \dfrac{0}{0}$ ← indeterminate form

STEP 2: $\dfrac{d}{dx}(x^3) = 3x^2$ $\quad \dfrac{d}{dx}(-2x + \sin 2x) = -2 + 2\cos 2x$

STEP 3: $\lim_{x \to 0} \dfrac{3x^2}{-2 + 2\cos 2x} = \dfrac{3(0)^2}{-2 + 2\cos 0} = \dfrac{0}{0}$ ← indeterminate form

STEP 4: That limit is still an indeterminate form, so proceed to STEP 5.

STEP 5: $\lim_{x \to 0} \dfrac{6x}{-4\sin 2x} = \dfrac{6(0)}{-4\sin 0} = \dfrac{0}{0}$

$\dfrac{d}{dx}(3x^2) = 6x$

$\dfrac{d}{dx}(-2 + 2\cos 2x) = -4\sin 2x$

That's still an indeterminate form, so repeat again:

$\lim_{x \to 0} \dfrac{6}{-8\cos 2x} = \dfrac{6}{-8\cos 0} = -\dfrac{3}{4}$

$\dfrac{d}{dx}(6x) = 6$

$\dfrac{d}{dx}(-4\sin 2x) = -8\cos 2x$

And that limit exists, so

$$\boxed{\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = -\frac{3}{4}}$$

## Limits Using a Maclaurin Series

### How do I evaluate a limit using Maclaurin series?

- Limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ or $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ may sometimes be evaluated by using Maclaurin series
- Usually this will be in a situation where attempting to evaluate the limit in the usual way returns an **indeterminate form** $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.
- In such a case:
  - STEP 1: Find the Maclaurin series for $f(x)$ and $g(x)$
  - STEP 2: Rewrite $\frac{f(x)}{g(x)}$ using the Maclaurin series in the numerator and denominator
  - STEP 3: Use algebra to simplify your new expression for $\frac{f(x)}{g(x)}$ as far as possible
  - STEP 4: Evaluate the limit using your simplified form of the expression



### Worked example

Use Maclaurin series to evaluate the limit

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x}$$

| Maclaurin series for special functions | $e^x = 1 + x + \frac{x^2}{2!} + \dots$ $\quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ <br> $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ $\quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ <br> $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ |
|---|---|

from exam formula booklet

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = \frac{0^3}{-2(0) + \sin 0} = \frac{0}{0}$$

indeterminate form

STEP 1: $-2x + \sin 2x = -2x + \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots\right)$

$$= -\frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots$$

STEP 2: $\frac{x^3}{-2x + \sin 2x} = \frac{x^3}{-\frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots}$

STEP 3: $\frac{x^3}{-\frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots} \times \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{1}{-\frac{4}{3} + \frac{4}{15}x^2 - \dots}$

STEP 4: $\lim_{x \to 0} \frac{1}{-\frac{4}{3} + \frac{4}{15}x^2 - \dots} = \frac{1}{-\frac{4}{3} + \frac{4}{15}(0)^2 - \dots} = \frac{1}{-\frac{4}{3}} = -\frac{3}{4}$

higher powers of x will also be zero

$$\boxed{\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = -\frac{3}{4}}$$