DP IB Maths: AA HL

5.11 MacLaurin Series

Contents

5.11.1 Maclaurin Series

Maclaurin Series of Standard Functions

What is a Maclaurin Series?

- A Maclaurin series is a way of representing a function as an infinite sum of increasing integer powers of x (x^1, x^2, x^3, etc.)
  - If all of the infinite number of terms are included, then the Maclaurin series is exactly equal to the original function
  - If we **truncate** (i.e., shorten) the Maclaurin series by stopping at some particular power of x, then the Maclaurin series is only an approximation of the original function
- A truncated Maclaurin series will always be exactly equal to the original function for $x = 0$
- In general, the approximation from a truncated Maclaurin series becomes less accurate as the value of x moves further away from zero
- The accuracy of a truncated Maclaurin series approximation can be improved by including more terms from the complete infinite series
  - So, for example, a series truncated at the x^7 term will give a more accurate approximation than a series truncated at the x^3 term

How do I find the Maclaurin series of a function 'from first principles'?

- Use the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

- This formula is in your exam formula booklet
- STEP 1: Find the values of $f(0)$, $f'(0)$, $f''(0)$, etc. for the function
  - An exam question will specify how many terms of the series you need to calculate (for example, "up to and including the term in x^4")
  - You may be able to use your GDC to find these values directly without actually having to find all the necessary derivatives of the function first
- STEP 2: Put the values from Step 1 into the general Maclaurin series formula
- STEP 3: Simplify the coefficients as far as possible for each of the powers of x

Is there an easier way to find the Maclaurin series for standard functions?

- Yes there is!
- The following Maclaurin series expansions of standard functions are contained in your exam formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

- Unless a question specifically asks you to derive a Maclaurin series using the general Maclaurin series formula, you can use those standard formulae from the exam formula booklet in your working

Is there a connection Maclaurin series expansions and binomial theorem series expansions?

- Yes there is!
- For a function like $(1+x)^n$ the binomial theorem series expansion is **exactly the same** as the Maclaurin series expansion for the same function
  - So unless a question specifically tells you to use the general Maclaurin series formula, you can use the binomial theorem to find the Maclaurin series for functions of that type
  - Or if you've forgotten the binomial series expansion formula for $(1+x)^n$ where n is not a positive integer, you can find the binomial theorem expansion by using the general Maclaurin series formula to find the Maclaurin series expansion

Worked example

a) Use the Maclaurin series formula to find the Maclaurin series for $f(x) = \sqrt{1+2x}$ up to and including the term in x^4.

$f(x) = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$

STEP 1: $f(0) = 1 \quad f'(0) = 1 \quad f''(0) = -1$

$f'''(0) = 3 \quad f^{(4)}(0) = -15$

STEP 2: $f(x) = 1 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(3) + \frac{x^4}{4!}(-15) + \ldots$

STEP 3: Up to the x^4 term,

$$\boxed{\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4}$$

Note: This is the same as the binomial theorem expansion of $(1+2x)^{\frac{1}{2}}$

b) Use your answer from part (a) to find an approximation for the value of $\sqrt{1.02}$, and compare the approximation found to the actual value of the square root.

Up to the x^4 term,

$$\boxed{\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4}$$ from part (a)

Let $x = 0.01$. Then $\sqrt{1+2x} = \sqrt{1+2(0.01)} = \sqrt{1.02}$.

So

$$\sqrt{1.02} \approx 1 + (0.01) - \frac{1}{2}(0.01)^2 + \frac{1}{2}(0.01)^3 - \frac{5}{8}(0.01)^4$$

$$\boxed{\sqrt{1.02} \approx 1.00995049375}$$

The exact value of the square root is

$$\sqrt{1.02} = 1.00995049383 6\ldots$$

The approximation is accurate to 10 d.p. or 11 s.f.

Maclaurin Series of Composites & Products

How can I find the Maclaurin series for a composite function?

- A **composite function** is a 'function of a function' or a 'function within a function'
  - For example sin(2x) is a composite function, with 2x as the 'inside function' which has been put into the simpler 'outside function' sin x
  - Similarly e^{x^2} is a composite function, with x^2 as the 'inside function' and e^x as the 'outside function'
- To find the Maclaurin series for a composite function:
  - STEP 1: Start with the Maclaurin series for the basic 'outside function'
    - Usually this will be one of the 'standard functions' whose Maclaurin series are given in the exam formula booklet
  - STEP 2: Substitute the 'inside function' every place that x appears in the Maclaurin series for the 'outside function'
    - So for sin(2x), for example, you would substitute 2x everywhere that x appears in the Maclaurin series for sin x
  - STEP 3: Expand the brackets and simplify the coefficients for the powers of x in the resultant Maclaurin series
- This method can theoretically be used for quite complicated 'inside' and 'outside' functions
  - On your exam, however, the 'inside function' will usually not be more complicated than something like kx (for some constant k) or x^n (for some constant power n)

How can I find the Maclaurin series for a product of two functions?

- To find the Maclaurin series for a product of two functions:
  - STEP 1: Start with the Maclaurin series of the individual functions
    - For each of these Maclaurin series you should only use terms up to an appropriately chosen power of x (see the worked example below to see how this is done!)
  - STEP 2: Put each of the series into brackets and multiply them together
    - Only keep terms in powers of x up to the power you are interested in
  - STEP 3: Collect terms and simplify coefficients for the powers of x in the resultant Maclaurin series

Worked example

a) Find the Maclaurin series for the function $f(x) = \ln(1+3x)$, up to and including the term in x^4.

Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

from exam formula booklet

STEP 1: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

STEP 2: $\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \ldots$

STEP 3: $\boxed{\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots}$

b) Find the Maclaurin series for the function $g(x) = e^x \sin x$, up to and including the term in x^4.

Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

STEP 1: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ ← Higher powers of x here will give powers higher than 4 when multiplied by the sin x series.

$\sin x = x - \frac{x^3}{6} + \ldots$ ← Don't need terms in powers of x higher than 4

STEP 2: $e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$

$$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^3}{6} - \frac{x^4}{6} \underbrace{- \cancel{\frac{x^5}{12}} - \cancel{\frac{x^6}{36}}}_{\text{Discard terms for powers higher than 4}}$$

Note that the x^4 terms cancel out

STEP 3: $\boxed{e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots}$

Differentiating & Integrating Maclaurin Series

How can I use differentiation to find Maclaurin Series?

- If you differentiate the Maclaurin series for a function *f*(x) term by term, you get the Maclaurin series for the function's derivative *f'*(x)
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of sin x is cos x
  - So if you differentiate the Maclaurin series for sin x term by term you will get the Maclaurin series for cos x

How can I use integration to find Maclaurin series?

- If you integrate the Maclaurin series for a derivative *f'*(x), you get the Maclaurin series for the function *f*(x)
  - Be careful however, as you will have a constant of integration to deal with
  - The value of the constant of integration will have to be chosen so that the series produces the correct value for *f*(0)
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of sin x is cos x
  - So if you integrate the Maclaurin series for cos x (and correctly deal with the constant of integration) you will get the Maclaurin series for sin x

Worked example

a) (i) Write down the derivative of $\arctan x$.

(ii) Hence use the Maclaurin series for $\arctan x$ to derive the Maclaurin series for $\frac{1}{1+x^2}$.

Standard derivatives	
$\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$

Maclaurin series for special functions		
	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

(i) $\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}}$

(ii) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

$$\Rightarrow \frac{1}{1+x^2} = \frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$$

$$\boxed{\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots}$$

Note : This is the same as the binomial theorem expansion of $(1+x^2)^{-1}$

b) (i) Write down the derivative of $-\sin x$.

(ii) Hence derive the Maclaurin series for $\cos x$, being sure to justify your method.

Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

(i) $\boxed{-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots}$

(ii) $-\sin x$ is the derivative of $\cos x$, so we can integrate the Maclaurin series for $-\sin x$ to find the Maclaurin series for $\cos x$.

$$\cos x = \int \left(-x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots\right) dx$$

constant of integration

$$= c - \frac{1}{2}x^2 + \frac{1}{4}\cdot\frac{x^4}{3!} - \frac{1}{6}\cdot\frac{x^6}{5!} + \frac{1}{8}\cdot\frac{x^8}{7!} - \ldots$$

$$= c - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

And $\cos(0) = 1$, so $c - \frac{0^2}{2!} + \frac{0^4}{4!} - \frac{0^6}{6!} + \frac{0^8}{8!} - \ldots = 1 \implies c = 1$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots}$$

5.11.2 Maclaurin Series from Differential Equations

Maclaurin Series for Differential Equations

Can I apply Maclaurin Series to solving differential equations?

- If you have a differential equation of the form $\frac{dy}{dx} = g(x,y)$ along with the value of $y(0)$ it is possible to build up the Maclaurin series of the solution $y = f(x)$ term by term
  - This does not necessarily tell you the explicit function of x that corresponds to the Maclaurin series you are finding
  - But the Maclaurin series you find is the exact Maclaurin series for the solution to the differential equation
- The Maclaurin series can be used to approximate the value of the solution *y* = *f*(*x*) for different values of x
  - You can increase the accuracy of this approximation by calculating additional terms of the Maclaurin series for higher powers of x

How can I find the Maclaurin Series for the solution to a differential equation?

- STEP 1: Use **implicit differentiation** to find expressions for y'', y''' etc., in terms of x, y and lower-order derivatives of y
  - The number of derivatives you need to find depends on how many terms of the Maclaurin series you want to find
  - For example, if you want the Maclaurin series up to the term, then you will need to find derivatives up to $y^{(4)}$ (the fourth derivative of y)
- STEP 2: Using the given initial value for $y(0)$, find the values of $y'(0)$, $y''(0)$, $y'''(0)$, etc., one by one
  - Each value you find will then allow you to find the value for the next higher derivative
- STEP 3: Put the values found in STEP 2 into the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$$

  - This formula is in your exam formula booklet
  - $y = f(x)$ is the solution to the differential equation, so $y(0)$ corresponds to $f(0)$ in the formula, $y'(0)$ corresponds to $f'(0)$, and so on
- STEP 4: Simplify the coefficients for each of the powers of x in the resultant Maclaurin series

Worked example

Consider the differential equation $y' = y^2 - x$ with the initial condition $y(0) = 2$.

a) Use implicit differentiation to find expressions for y'', y''' and $y^{(4)}$.

STEP 1 :

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - x) = 2yy' - 1$$

$$\boxed{y'' = 2yy' - 1}$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(2yy' - 1) = 2yy'' + 2(y')^2$$

$$\boxed{y''' = 2yy'' + 2(y')^2}$$

$$y^{(4)} = \frac{d}{dx}(y''') = \frac{d}{dx}(2yy'' + 2(y')^2)$$

$$= 2y'y'' + 2yy''' + 4y'y''$$

$$\boxed{y^{(4)} = 6y'y'' + 2yy'''}$$

b) Use the given initial condition to find the values of $y'(0)$, $y''(0)$, $y'''(0)$ and $y^{(4)} = 0$.

STEP 2:

$y(0) = 2$, so $y'(0) = 2^2 - 0 = 4$ $\quad y' = y^2 - x$

Then $y''(0) = 2(2)(4) - 1 = 15$ $\quad y'' = 2yy' - 1$

$y'''(0) = 2(2)(15) + 2(4)^2 = 92$ $\quad y''' = 2yy'' + 2(y')^2$

$y^{(4)}(0) = 6(4)(15) + 2(2)(92) = 728$ $\quad y^{(4)} = 6y'y'' + 2yy'''$

$$\boxed{\begin{array}{ll} y'(0) = 4 & y''(0) = 15 \\ y'''(0) = 92 & y^{(4)}(0) = 728 \end{array}}$$

Let $y = f(x)$ be the solution to the differential equation with the given initial condition.

c) Find the first five terms of the Maclaurin series for $f(x)$.

Maclaurin series $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$

from exam formula booklet

STEP 3: $f(x) = 2 + x(4) + \frac{x^2}{2!}(15) + \frac{x^3}{3!}(92) + \frac{x^4}{4!}(728) + \dots$

STEP 4:

$$\boxed{f(x) = 2 + 4x + \frac{15}{2}x^2 + \frac{46}{3}x^3 + \frac{91}{3}x^4 + \dots}$$