DP IB Maths: AA HL

5.1 Differentiation

Contents

5.1.1 Introduction to Differentiation

Introduction to Derivatives

- Before introducing a **derivative**, an understanding of a **limit** is helpful

What is a limit?

- The **limit** of a **function** is the value a function (of x) approaches as x approaches a particular value from either side
  - Limits are of interest when the function is undefined at a particular value
  - For example, the function $f(x) = \frac{x^4 - 1}{x - 1}$ will approach a limit as x approaches 1 from both below and above but is undefined at $x = 1$ as this would involve dividing by zero

What might I be asked about limits?

- You may be asked to predict or estimate limits from a table of function values or from the graph of $y = f(x)$
- You may be asked to use your GDC to plot the graph and use values from it to estimate a limit

What is a derivative?

- **Calculus** is about **rates** of **change**
  - the way a car's position on a road changes is its speed (velocity)
  - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with x
- The **derivative** of a function is a function that relates the **gradient** to the value of x
- The derivative is also called the **gradient function**

How are limits and derivatives linked?

- Consider the point P on the graph of $y = f(x)$ as shown below
  - $[PQ_i]$ is a series of chords

- The **gradient** of the **function** $f(x)$ at the point P is **equal** to the **gradient** of the **tangent** at point P
- The **gradient** of the **tangent** at point P is the **limit** of the **gradient** of the chords $[PQ_i]$ as point Q 'slides' down the curve and gets ever closer to point P
- The **gradient** of the function changes as x changes
- The **derivative** is the function that calculates the gradient from the value x

What is the notation for derivatives?

- For the function $y = f(x)$, the **derivative**, with respect to x, would be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

- Different variables may be used
  - e.g. If $V = f(s)$ then $\frac{\mathrm{d}V}{\mathrm{d}s} = f'(s)$

Worked example

The graph of $y = f(x)$ where $f(x) = x^3 - 2$ passes through the points $P(2, 6)$, $A(2.3, 10.167)$, $B(2.1, 7.261)$ and $C(2.05, 6.615125)$.

a) Find the gradient of the chords $[PA]$, $[PB]$ and $[PC]$.

Gradient of a line (chord) is " $\frac{y_2 - y_1}{x_2 - x_1}$ "

$[PA]$: $\frac{10.167 - 6}{2.3 - 2} = 13.89$

$[PB]$: $\frac{7.261 - 6}{2.1 - 2} = 12.61$

$[PC]$: $\frac{6.615125 - 6}{2.05 - 2} = 12.3$

Gradient of chords are: $[PA]$ 13.89, $[PB]$ 12.61, $[PC]$ 12.3025

b) Estimate the gradient of the tangent to the curve at the point P.

There will be a limit the gradient of the chord reaches as the difference in the x-coordinates approaches zero.

Estimate of gradient of tangent at $x = 2$ is 12

Differentiating Powers of x

What is differentiation?

- **Differentiation** is the process of finding an expression of the **derivative** (**gradient function**) from the expression of a function

How do I differentiate powers of x?

- **Powers** of x are **differentiated** according to the following formula:
  - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{Q}$
  - This is given in the **formula booklet**
- If the power of x is **multiplied** by a **constant** then the derivative is also multiplied by that constant
  - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$ where $n \in \mathbb{Q}$ and a is a constant
- The **alternative notation** (to $f'(x)$) is to use $\frac{\mathrm{d}y}{\mathrm{d}x}$
  - If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$
    - e.g. If $y = -4x^{\frac{1}{2}}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -4 \times \frac{1}{2} \times x^{\frac{1}{2}-1} = -2x^{-\frac{1}{2}}$
- Don't forget these **two** special cases:
  - If $f(x) = ax$ then $f'(x) = a$
    - e.g. If $y = 6x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$
  - If $f(x) = a$ then $f'(x) = 0$
    - e.g. If $y = 5$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
  - These allow you to differentiate **linear terms** in x and **constants**
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
  - e.g. If $f(x) = 2\sqrt{x}$ then rewrite as $f(x) = 2x^{\frac{1}{2}}$ and differentiate
- Functions involving **fractions** with **denominators** in terms of x will need to be rewritten as **negative powers** of x first
  - e.g. If $f(x) = \frac{4}{x}$ then rewrite as $f(x) = 4x^{-1}$ and differentiate

How do I differentiate sums and differences of powers of x?

- The formulae for differentiating powers of x apply to **all rational** powers so it is possible to differentiate any expression that is a **sum** or **difference** of **powers** of x
  - e.g. If $f(x) = 5x^4 - 3x^{\frac{2}{3}} + 4$ then

    $$f'(x) = 5 \times 4x^{4-1} - 3 \times \frac{2}{3}x^{\frac{2}{3}-1} + 0$$

    $$f'(x) = 20x^3 - 2x^{-\frac{1}{3}}$$
- **Products** and **quotients cannot** be differentiated in this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = (2x-3)(x^2-4)$ then expand to $f(x) = 2x^3 - 3x^2 - 8x + 12$ which **is** a **sum/difference** of powers of x and can be differentiated

Worked example

The function $f(x)$ is given by

$$f(x) = 2x^3 + \frac{4}{\sqrt{x}}, \text{ where } x > 0$$

Find the derivative of $f(x)$

Rewrite f(x) so every term is a power of x

$f(x) = 2x^3 + 4x^{-1/2}$

Differentiate by applying the formula

$f'(x) = 6x^2 - 2x^{-3/2}$

anx^{n-1}

take care with negatives

$-\frac{1}{2} - 1 = -\frac{3}{2}$

$$\boxed{\therefore\ f'(x) = 6x^2 - 2x^{-3/2}}$$

5.1.2 Applications of Differentiation

Finding Gradients

How do I find the gradient of a curve at a point?

- The **gradient of a curve** at a point is the **gradient of the tangent** to the curve at that point
- **Find the gradient** of a curve at a point by **substituting the value of x** at that point into the curve's **derivative function**
- For example, if $f(x) = x^2 + 3x - 4$
  - then $f'(x) = 2x + 3$
  - and the gradient of $y = f(x)$ when $x = 1$ is $f'(1) = 2(1) + 3 = 5$
  - and the gradient of $y = f(x)$ when $x = -2$ is $f'(-2) = 2(-2) + 3 = -1$
- Although your GDC won't find a derivative function for you, it is possible to use your **GDC** to **evaluate the derivative** of a function at a point, using $\frac{d}{dx}(\square)_{x=\square}$

Worked example

A function is defined by $f(x) = x^3 + 6x^2 + 5x - 12$.

(a) Find $f'(x)$.

Find $f'(x)$ by differentiating

$f'(x) = 3x^2 + 2 \times 6x^1 + 5x^0$

$\boxed{f'(x) = 3x^2 + 12x + 5}$

(b) Hence show that the gradient of $y = f(x)$ when $x = 1$ is 20.

Substitute $x = 1$ into $f'(x)$

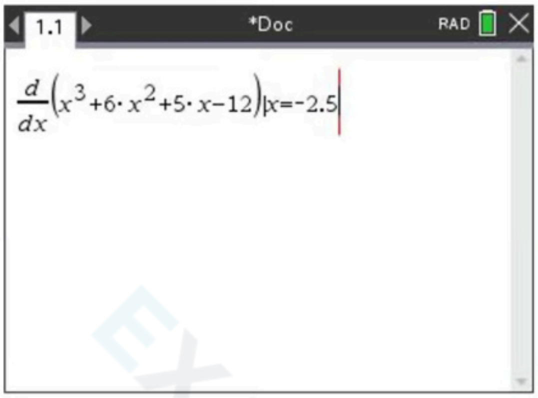
$f'(1) = 3(1)^2 + 12(1) + 5$

$= 3 + 12 + 5$

$\boxed{f'(1) = 20}$

(c) Find the gradient of $y = f(x)$ when $x = -2.5$.

Use the GDC to evaluate the derivative of $f(x)$ at $x = -2 \cdot 5$

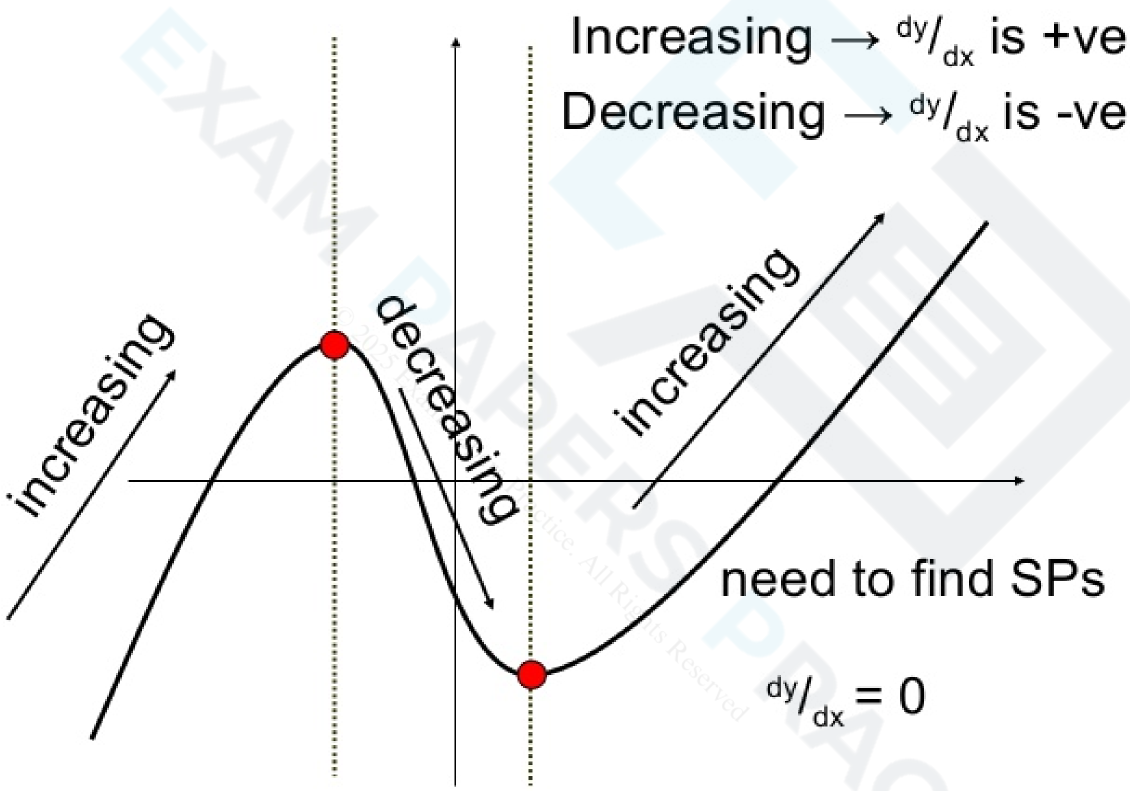


$\boxed{f'(-2 \cdot 5) = -6 \cdot 25}$

Increasing & Decreasing Functions

What are increasing and decreasing functions?

- A function, $f(x)$, is **increasing** if $\boldsymbol{f'(x) > 0}$
  - This means the **value** of the **function** ('output') **increases** as x **increases**
- A function, $f(x)$, is **decreasing** if $\boldsymbol{f'(x) < 0}$
  - This means the **value** of the **function** ('output') **decreases** as x **increases**
- A function, $f(x)$, is **stationary** if $\boldsymbol{f'(x) = 0}$



How do I find where functions are increasing, decreasing or stationary?

- To identify the **intervals** on which a function is increasing or decreasing

  STEP 1
  Find the derivative **f'(*x*)**

  STEP 2
  Solve the inequalities

$f'(x) > 0$ (for increasing intervals) and/or
$f'(x) < 0$ (for decreasing intervals)

- Most functions are a combination of **increasing, decreasing** and **stationary**
  - a range of values of x (**interval**) is given where a function satisfies each condition
  - e.g. The function $f(x) = x^2$ has **derivative** $f'(x) = 2x$ so
    - $f(x)$ is **decreasing** for $x < 0$
    - $f(x)$ is **stationary** at $x = 0$
    - $f(x)$ is **increasing** for $x > 0$

Worked example

$f(x) = x^2 - x - 2$

a) Determine whether $f(x)$ is increasing or decreasing at the points where $x = 0$ and $x = 3$.

Differentiate

$f'(x) = 2x - 1$

At $x = 0$, $f'(0) = 2 \times 0 - 1 = -1 < 0$ $\therefore$ decreasing

At $x = 3$, $f'(3) = 2 \times 3 - 1 = 5 > 0$ $\therefore$ increasing

$\therefore$ At $x = 0$, $f(x)$ is decreasing
At $x = 3$, $f(x)$ is increasing

b) Find the values of x for which $f(x)$ is an increasing function.

$f(x)$ is increasing when $f'(x) > 0$

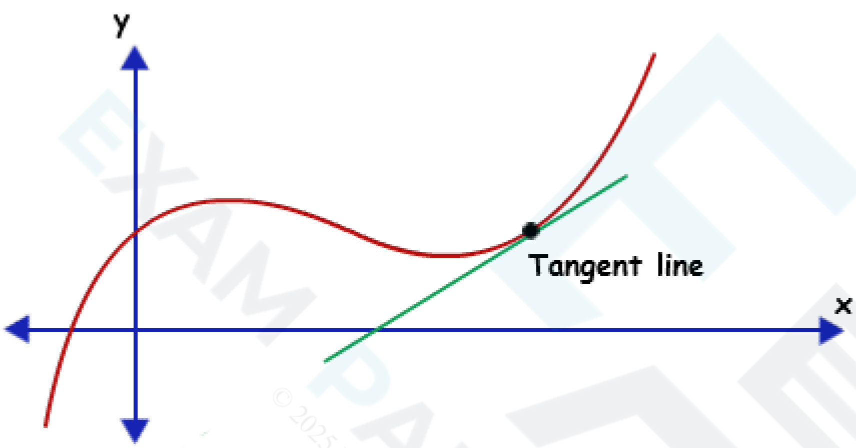
$f'(x) > 0$
$2x - 1 > 0$
$x > \frac{1}{2}$

$\therefore$ $f(x)$ is increasing for $x > \frac{1}{2}$

Tangents & Normals

What is a tangent?

- At any point on the graph of a (non-linear) **function**, the **tangent** is the straight line that **touches** the graph at a point **without crossing** through it
- Its **gradient** is given by the **derivative function**

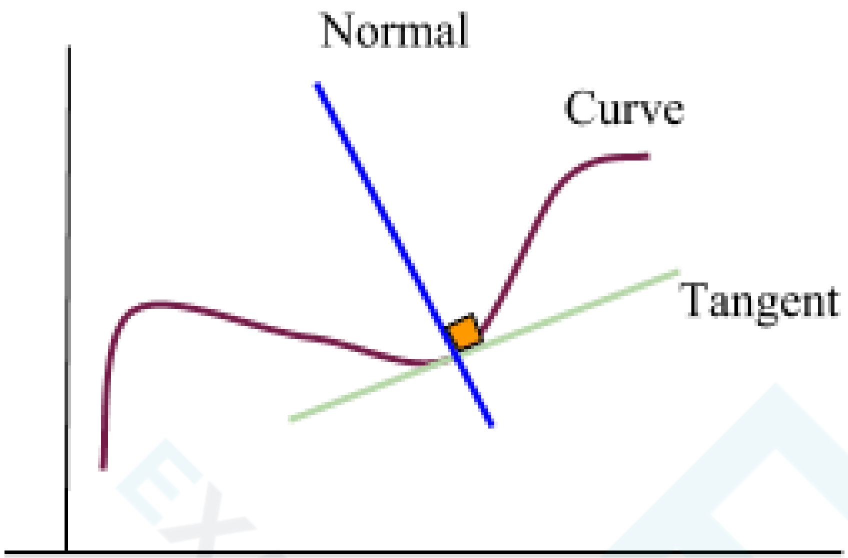


How do I find the equation of a tangent?

- To find the **equation of a straight line**, a **point** and the **gradient** are needed
- The **gradient**, m, of the **tangent** to the function $y = f(x)$ at (x_1, y_1) is $\boldsymbol{f'(x_1)}$
- Therefore find the **equation** of the **tangent** to the function $y = f(x)$ at the point (x_1, y_1) by substituting the gradient, $f'(x_1)$, and point (x_1, y_1) into $y - y_1 = m(x - x_1)$, giving:
  - $y - y_1 = \boldsymbol{f'(x_1)}(x - x_1)$
- (You could also substitute into $y = mx + c$ but it is usually quicker to substitute into $y - y_1 = m(x - x_1)$)

What is a normal?

- At any point on the graph of a (non-linear) function, the **normal** is the straight line that passes through that point and is **perpendicular** to the **tangent**



How do I find the equation of a normal?

- The **gradient** of the **normal** to the function $y = f(x)$ at (x_1, y_1) is $\dfrac{-1}{f'(x_1)}$
- Therefore find the **equation** of the **normal** to the function $y = f(x)$ at the point (x_1, y_1) by using

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

Worked example

The function $f(x)$ is defined by

$$f(x) = 2x^4 + \frac{3}{x^2} \qquad x \neq 0$$

a) Find an equation for the tangent to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $y = mx + c$.

First find $f'(x)$ by differentiating

$f(x) = 2x^4 + 3x^{-2}$ Rewrite as powers of x

$f'(x) = 8x^3 - 6x^{-3}$

For a tangent, "$y - y_1 = f'(a)(x - x_1)$"

At $x = 1$, $y = 2(1)^4 + \frac{3}{(1)^2} = 5$

$f'(1) = 8(1)^3 - \frac{6}{(1)^3} = 2$

$\therefore\ y - 5 = 2(x - 1)$

$\boxed{\text{Tangent at } x = 1 \text{, is } y = 2x + 3}$

b) Find an equation for the normal at the point where $x = 1$, giving your answer in the form $ax + by + d = 0$, where a, b and d are integers.

For a normal, "$y - y_1 = \frac{-1}{f'(a)}(x - x_1)$"

Using results from part (a):

$$y - 5 = \frac{-1}{2}(x - 1)$$

$$y = -\frac{1}{2}x + \frac{11}{2}$$

$$2y = -x + 11$$

$\therefore$ Equation of normal is $x + 2y - 11 = 0$