



5.1 Differentiation

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5.1.1 Introduction to Differentiation

Introduction to Derivatives

• Before introducing a **derivative**, an understanding of a **limit** is helpful

What is a limit?

- The **limit** of a **function** is the value a function (of *X*) approaches as *X* approaches a particular value from either side
 - Limits are of interest when the function is undefined at a particular value
 - For example, the function $f(x) = \frac{x^4 1}{x 1}$ will approach a limit as X approaches 1 from both below and above but is undefined at X = 1 as this would involve dividing by zero

What might I be asked about limits?

- You may be asked to predict or estimate limits from a table of function values or from the graph of y = f(x)
- You may be asked to use your GDC to plot the graph and use values from it to estimate a limit

What is a derivative?

- Calculus is about rates of change
 - the way a car's position on a road changes is its speed
 - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with *X*
- The **derivative** of a function is a function that relates the gradient to the value of X
- It is also called the gradient function

What is the notation for derivatives?

For the function y = f(x) the **derivative**, with respect to X, would be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

Different variables may be used

• e.g. If
$$V = f(s)$$
 then $\frac{\mathrm{d} V}{\mathrm{d} s} = f'(s)$



The graph of y = f(x) where $f(x) = x^3 - 2$ passes through the points P(2, 6), A(2.3, 10.167), B(2.1, 7.261) and C(2.05, 6.615125).

a) Find the gradient of the chords [PA], [PB] and [PC].

$$[PA]$$
: $10 \cdot 167 - 6$ = 13.89

$$[PB]: \frac{7.261-6}{2.1-2} = 12.61$$

b) Estimate the gradient of the tangent to the curve at the point ${\it P}$.

There will be a limit the gradient of the chord reaches as the difference in the x-coordinates approaches zero.

Estimate of gradient of tangent at
$$x=2$$
 is 12



Differentiating Powers of x

What is differentiation?

• **Differentiation** is the process of finding an expression of the **derivative** (**gradient function**) from the expression of a function

How do I differentiate powers of x?

- Powers of X are differentiated according to the following formula:
 - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{Z}$
 - This is given in the formula booklet
- If the power of X is multiplied by a constant then the derivative is also multiplied by that constant
 - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$ where $n \in \mathbb{Z}$ and a is a constant
- The alternative notation (to f'(x)) is to use $\frac{\mathrm{d}y}{\mathrm{d}x}$

If
$$y = ax^n$$
 then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$

• e.g. If
$$y = -4x^5$$
 then $\frac{dy}{dx} = -4 \times 5x^{5-1} = -20x^4$

■ Don't forget these **two** special cases:

• If
$$f(x) = ax$$
 then $f'(x) = a$

• e.g. If
$$y = 6x$$
 then $\frac{dy}{dx} = 6$

• If
$$f(x) = a$$
 then $f'(x) = 0$

• e.g. If
$$y = 5$$
 then $\frac{dy}{dx} = 0$

- These allow you to differentiate **linear terms** in *X* and **constants**
- Functions involving fractions with denominators in terms of X will need to be rewritten as negative powers of X first

If
$$f(x) = \frac{4}{x}$$
 then rewrite as $f(x) = 4x^{-1}$ and differentiate

How do I differentiate sums and differences of powers of x?

 The formulae for differentiating powers of X apply to all integer powers so it is possible to differentiate any expression that is a sum or difference of powers of X

• e.g. If
$$f(x) = 5x^4 + 2x^3 - 3x + 4$$
 then $f'(x) = 5 \times 4x^{4-1} + 2 \times 3x^{3-1} - 3 + 0$



$$f'(x) = 20x^3 + 6x^2 - 3$$

- Products and quotients cannot be differentiated in this way so would need expanding/simplifying
 - e.g. If $f(x) = (2x-3)(x^2-4)$ then expand to $f(x) = 2x^3 3x^2 8x + 12$ which is a **sum/difference** of powers of *X* and can be differentiated

The function f(x) is given by

$$f(x) = x^3 - 2x^2 + 3 - \frac{4}{x^3}$$

Find the derivative of f(x)

Rewrite f(x) so every term is a power of a

$$f(x) = x^3 - 2x^2 + 3 - 4x^{-3}$$

Differentiate by applying the formula (3 is a special case) $f'(x) = 3x^2 - 4x + 12x^{-14}$

$$n \infty^{n-1}$$
 \int \int $a n \infty^{n-1}$

$$f'(x) = 3x^2 - 4x + \frac{12}{x^4}$$



5.1.2 Applications of Differentiation

Finding Gradients

How do I find the gradient of a curve at a point?

- The gradient of a curve at a point is the gradient of the tangent to the curve at that point
- Find the gradient of a curve at a point by substituting the value of **X** at that point into the curve's derivative function
- For example, if $f(x) = x^2 + 3x 4$
 - then f'(x) = 2x + 3
 - and the gradient of y = f(x) when x = 1 is f'(1) = 2(1) + 3 = 5
 - and the gradient of y = f(x) when x = -2 is f'(-2) = 2(-2) + 3 = -1
- Although your GDC won't find a derivative function for you, it is possible to use your GDC to evaluate

the derivative of a function at a point, using $\frac{d}{dx}()_{x=0}$



A function is defined by $f(x) = x^3 + 6x^2 + 5x - 12$.

(a) Find f'(x).

$$f'(x) = 3x^2 + 2 \times 6x^1 + 5x^0$$

$$f'(x) = 3x^2 + 12x + 5$$

(b) Hence show that the gradient of y = f(x) when x = 1 is 20.

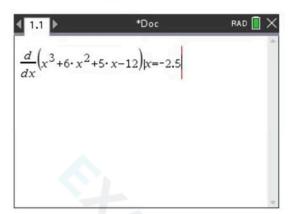
Substitute
$$x = 1$$
 into $f'(x)$

$$f'(1) = 3(1)^2 + 12(1) + 5$$

(c) Find the gradient of y = f(x) when x = -2.5.



Use the GDC to evaluate the derivative of f(x) at x = -2.5



$$f'(-2\cdot 5) = -6\cdot 25$$



Increasing & Decreasing Functions

What are increasing and decreasing functions?

- A function, f(x), is increasing if f'(x) > 0
 - This means the value of the function ('output') increases as x increases
- A function, f(x), is **decreasing** if **f'(x) < 0**
 - This means the value of the function ('output') decreases as x increases
- A function, f(x), is stationary if f'(x) = 0

How do I find where functions are increasing, decreasing or stationary?

- To identify the intervals on which a function is increasing or decreasing
 STEP1 Find the derivative f'(x)
 STEP2 Solve the inequalities f'(x) > 0 (for increasing intervals) and/or f'(x) < 0 (for decreasing intervals)</p>
- Most functions are a combination of increasing, decreasing and stationary
 - a range of values of x (interval) is given where a function satisfies each condition
 - e.g. The function $f(x) = x^2$ has **derivative** f'(x) = 2x so
 - f(x) is decreasing for x < 0
 - f(x) is stationary at x = 0
 - f(x) is increasing for x > 0



$$f(x) = x^2 - x - 2$$

a) Determine whether f(x) is increasing or decreasing at the points where x = 0 and x = 3.

```
Differentiate
f(x) = 2x - 1
At x = 0, f'(0) = 2x0 - 1 = -1 < 0 is decreosing
At x = 3, f'(3) = 2x3 - 1 = 5 > 0 increasing

At x = 0, f(x) is decreasing

At x = 3, f(x) is increasing
```

b) Find the values of X for which f(x) is an increasing function.

```
f(x) is increasing when f'(x) > 0

f'(x) > 0

2x - 1 > 0

x > \frac{1}{2}

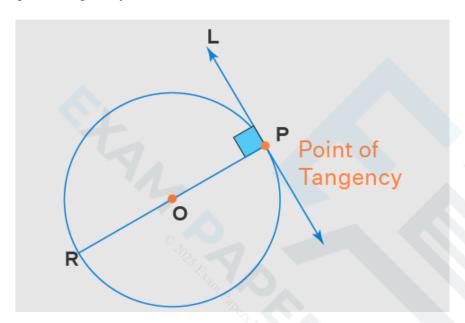
f(x) is increasing for x > \frac{1}{2}
```



Tangents & Normals

What is a tangent?

- At any point on the graph of a (non-linear) **function**, the **tangent** is the straight line that **touches** the graph at a point **without crossing** through it
- Its gradient is given by the derivative function



How do I find the equation of a tangent?

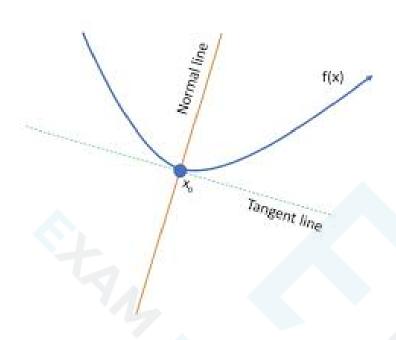
- To find the **equation of a straight line**, a **point** and the **gradient** are needed
- The gradient, m, of the tangent to the function y = f(x) at (x_1, y_1) is $f'(x_1)$
- Therefore find the **equation** of the **tangent** to the function y = f(x) at the point (x_1, y_1) by substituting the gradient, $f'(x_1)$, and point (x_1, y_1) into $y y_1 = m(x x_1)$, giving: $y y_1 = f'(x_1)(x x_1)$
- (You could also substitute into y = mx + c but it is usually quicker to substitute into $y y_1 = m(x x_1)$)

What is a normal?

 At any point on the graph of a (non-linear) function, the normal is the straight line that passes through that point and is perpendicular to the tangent



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How do I find the equation of a normal?

The gradient of the normal to the function y = f(x) at (x_1, y_1) is $\frac{-1}{f'(x_1)}$

Therefore find the **equation** of the **normal** to the function y = f(x) at the point (x_1, y_1) by using

$$y-y_1 = \frac{-1}{f'(x_1)}(x-x_1)$$



The function f(x) is defined by

$$f(x) = 2x^4 + \frac{3}{x^2}$$
 $x \neq 0$

Find an equation for the tangent to the curve y = f(x) at the point where x = 1, giving your answer in the form y = mx + c.

First find
$$f'(x)$$
 by differentiating

$$f(x) = 2x^{14} + 3x^{-2}$$
Rewrite as powers of x

$$f'(x) = 8x^{3} - 6x^{-3}$$

For a tangent, "y-y₁ = $f(a)(x-x_{1})$ "

At $x = 1$, $y = 2(1)^{14} + \frac{3}{11^{3}} = 5$

$$f'(1) = 8(1)^{3} - \frac{6}{(1)^{3}} = 2$$

$$\therefore y - 5 = 2(x-1)$$

Tangent at x=1, is y=2x+3

Find an equation for the normal at the point where x=1, giving your answer in the form ax+by+d=0, where a, b and d are integers.



For a normal, "y-y₁ =
$$\frac{-1}{f'(a)}(x-x_1)$$
"

Using results from part (a):

y-5 = $\frac{-1}{2}(x-1)$

y = $-\frac{1}{2}x + \frac{11}{2}$

" Equation of normal is
$$x+2y-11=0$$



Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local minimum and maximum points are two types of stationary point
 - The **gradient function** (derivative) at such points equals zero

i.e.
$$f'(x) = 0$$

- A local minimum point, (X, f(X)) will be the lowest value of f(X) in the local vicinity of the value of X
 - The function may reach a **lower** value further afield
- Similarly, a local maximum point, (X, f(X)) will be the greatest value of f(X) in the local vicinity of the value of X
 - The function may reach a **greater** value further afield
- The graphs of many functions tend to infinity for large values of X
 (and/or minus infinity for large negative values of X)
- The **nature** of a stationary point refers to whether it is a local **minimum** or local **maximum** point

How do I find the coordinates and nature of stationary points?

The instructions below describe how to find **local minimum** and **maximum points** using a **GDC** on the graph of the function y = f(x).

STEP 1

Plot the graph of y = f(x)

Sketch the graph as part of the solution

STEP 2

Use the options from the graphing screen to "solve for minimum"

The GDC will display the X and Y coordinates of the first minimum point

Scroll onwards to see there are anymore minimum points

Note down the coordinates and the type of stationary point

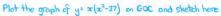
STEP 3

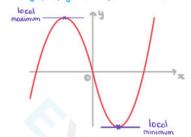
Repeat STEP 2 but use "solve for maximum" on your GDC

- In STEP 2 the nature of the stationary point should be easy to tell from the graph
 - a local minimum changes the function from decreasing to increasing
 - the gradient changes from **negative** to **positive**
 - a local maximum changes the function from increasing to decreasing
 - the gradient changes from positive to negative



Find the stationary points of $f(x) = x(x^2 - 27)$, and state their nature.





. Stationary points are

(3,-54) LOCAL MINIMUM POINT (-3,54) LOCAL MAXIMUM POINT



5.1.3 Modelling with Differentiation

Modelling with Differentiation

What can be modelled with differentiation?

- Recall that differentiation is about the rate of change of a function and provides a way of finding minimum and maximum values of a function
- Anything that involves maximising or minimising a quantity can be modelled using differentiation; for example
 - minimising the cost of raw materials in manufacturing a product
 - the maximum height a football could reach when kicked
- These are called **optimisation** problems

What modelling assumptions are used in optimisation problems?

- The quantity being **optimised** needs to be dependent on a **single** variable
 - If other variables are initially involved, constraints or assumptions about them will need to be made; for example
 - minimising the cost of the main raw material timber in manufacturing furniture say the cost of screws, glue, varnish, etc can be fixed or considered negligible
 - Other **modelling assumptions** may have to be made too; for example
 - ignoring air resistance and wind when modelling the path of a kicked football

How do I solve optimisation problems?

- In optimisation problems, letters other than X, Y and f are often used including capital letters
 - ullet V is often used for volume, S for surface area
 - If for radius if a circle, cylinder or sphere is involved
- Derivatives can still be found but be clear about which variable is independent (X) and which is dependent (Y)
 - a GDC may always use X and Y but ensure you use the correct variable throughout your working and final answer
- Problems often start by linking two connected quantities together for example volume and surface area
 - where more than one variable is involved, constraints will be given such that the quantity of interest can be rewritten in terms of a single variable
- Once the quantity of interest is written as a function of a single variable, differentiation can be used to maximise or minimise the quantity as required

STEP 1

Rewrite the quantity to be optimised as a single variable, using any constraints given in the question

STEP 2

Use your GDC to find the (local) maximum or minimum points as required



Plot the graph of the function and use the graphing features of the GDC to "solve for minimum/maximum" as required

STEP 3

Note down the solution from your GDC and interpret the answer(s) in the context of the question

Examiner Tip

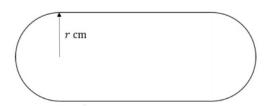
• The first part of rewriting a quantity as a single variable is often a "show that" question – this means you may still be able to access later parts of the question even if you can't do this bit







A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.



The total area of the bed is to be $100\,\pi\;m^2$.

a) Show that the perimeter of the bed is given by the formula

$$P = \pi \left(r + \frac{100}{r} \right)$$



The width of the rectangle is 2rm and its length Lm. The AREA of the bed, 1007 m² is given by

$$\therefore \pi r^{2} + 2rL = 100\pi$$

$$2rL = 100\pi - \pi r^{2}$$

$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$
Write L in terms of r

The PERIMETER of the bed is

Use L from the area constraint to write P in terms of ronly

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$

$$P = \pi r + \frac{100\pi}{r}$$



b) Find
$$\frac{\mathrm{d}P}{\mathrm{d}r}$$
 .

Rewrite P of powers of r

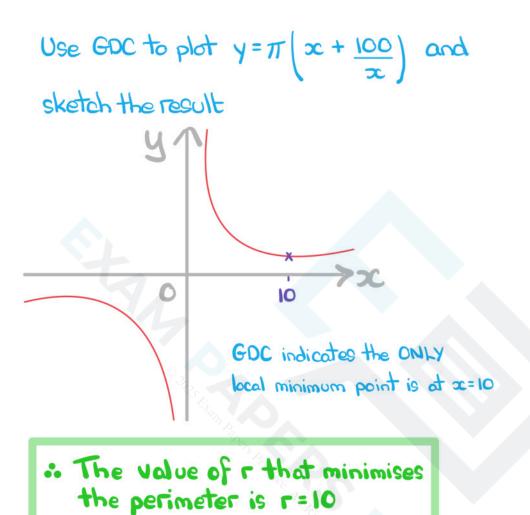
$$P = \pi \left(\Gamma + 100 \Gamma^{-1} \right)$$

$$\frac{dP}{dr} = \pi \left(1 - 100 \Gamma^{-2} \right)$$

$$\frac{dP}{dr} = \pi \left(1 - \frac{100}{r^2} \right)$$

c) Find the value of r that minimises the perimeter.





d) Hence find the minimum perimeter.



The minimum perimeter will be the y-coordinate of the local minimum point found in part (c) from GDC. y = 62.831.853... (when x = 10)

.. Minimum perimeter is 62.8 m (3 s.f.)