DP IB Maths: AI HL

5.1 Differentiation

Contents

5.1.1 Introduction to Differentiation

Introduction to Derivatives

- Before introducing a **derivative**, an understanding of a **limit** is helpful

What is a limit?

- The **limit** of a **function** is the value a function (of x) approaches as x approaches a particular value from either side
  - Limits are of interest when the function is undefined at a particular value
  - For example, the function $f(x) = \frac{x^4 - 1}{x - 1}$ will approach a limit as x approaches 1 from both below and above but is undefined at $x = 1$ as this would involve dividing by zero

What might I be asked about limits?

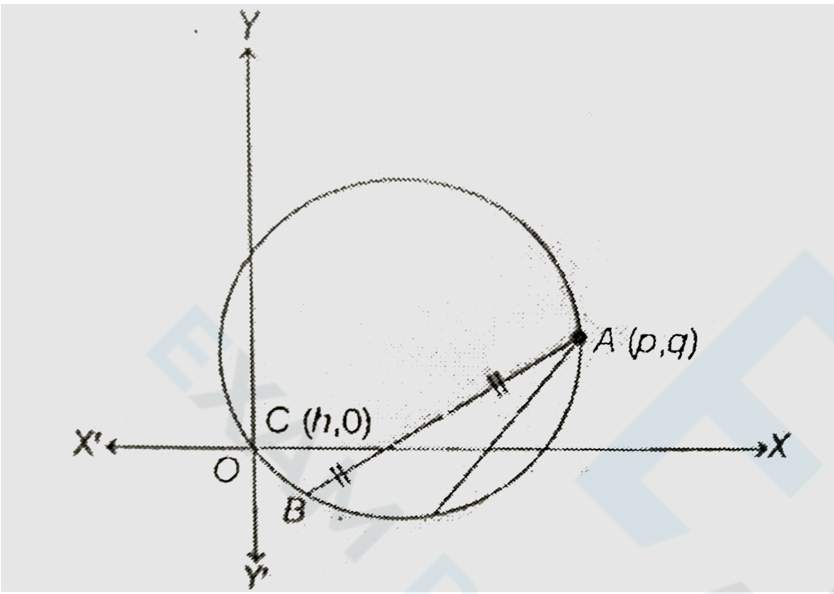
- You may be asked to predict or estimate limits from a table of function values or from the graph of $y = f(x)$
- You may be asked to use your GDC to plot the graph and use values from it to estimate a limit

What is a derivative?

- **Calculus** is about **rates** of **change**
  - the way a car's position on a road changes is its speed (velocity)
  - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with x
- The **derivative** of a function is a function that relates the **gradient** to the value of x
- The derivative is also called the **gradient function**

How are limits and derivatives linked?

- Consider the point P on the graph of $y = f(x)$ as shown below
  - $[PQ_i]$ is a series of chords



- The **gradient** of the **function** $f(x)$ at the point P is **equal** to the **gradient** of the **tangent** at point P
- The **gradient** of the **tangent** at point P is the **limit** of the **gradient** of the chords $[PQ_i]$ as point Q 'slides' down the curve and gets ever closer to point P
- The **gradient** of the function changes as x changes
- The **derivative** is the function that calculates the gradient from the value x

What is the notation for derivatives?

- For the function $y = f(x)$, the **derivative**, with respect to x, would be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

- Different variables may be used
  - e.g. If $V = f(s)$ then $\frac{\mathrm{d}V}{\mathrm{d}s} = f'(s)$

Worked example

The graph of $y = f(x)$ where $f(x) = x^3 - 2$ passes through the points $P(2,\ 6)$, $A(2.3,\ 10.167)$, $B(2.1,\ 7.261)$ and $C(2.05,\ 6.615125)$.

a) Find the gradient of the chords $[PA]$, $[PB]$ and $[PC]$.

Gradient of a line (chord) is " $\frac{y_2-y_1}{x_2-x_1}$ "

$[PA]$: $\frac{10.167-6}{2.3-2} = 13.89$

$[PB]$: $\frac{7.261-6}{2.1-2} = 12.61$

$[PC]$: $\frac{6.615125-6}{2.05-2} = 12.3$

Gradient of chords are: $[PA]$ 13.89
$[PB]$ 12.61
$[PC]$ 12.3025

b) Estimate the gradient of the tangent to the curve at the point P.

There will be a limit the gradient of the chord reaches as the difference in the x-coordinates approaches zero.

Estimate of gradient of tangent at $x = 2$ is 12

Differentiating Powers of x

What is differentiation?

- **Differentiation** is the process of finding an expression of the **derivative** (**gradient function**) from the expression of a function

How do I differentiate powers of x?

- **Powers** of x are **differentiated** according to the following formula:
  - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{Q}$
  - This is given in the **formula booklet**
- If the power of x is **multiplied** by a **constant** then the derivative is also multiplied by that constant
  - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$ where $n \in \mathbb{Q}$ and a is a constant
- The **alternative notation** (to $f'(x)$) is to use $\frac{\mathrm{d}y}{\mathrm{d}x}$
  - If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$
    - e.g. If $y = -4x^{\frac{1}{2}}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -4 \times \frac{1}{2} \times x^{\frac{1}{2}-1} = -2x^{-\frac{1}{2}}$
- Don't forget these **two** special cases:
  - If $f(x) = ax$ then $f'(x) = a$
    - e.g. If $y = 6x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$
  - If $f(x) = a$ then $f'(x) = 0$
    - e.g. If $y = 5$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
  - These allow you to differentiate **linear terms** in x and **constants**
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
  - e.g. If $f(x) = 2\sqrt{x}$ then rewrite as $f(x) = 2x^{\frac{1}{2}}$ and differentiate
- Functions involving **fractions** with **denominators** in terms of x will need to be rewritten as **negative powers** of x first
  - e.g. If $f(x) = \frac{4}{x}$ then rewrite as $f(x) = 4x^{-1}$ and differentiate

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How do I differentiate sums and differences of powers of x?

- The formulae for differentiating powers of x apply to **all rational** powers so it is possible to differentiate any expression that is a **sum** or **difference** of **powers** of x
  - e.g. If $f(x) = 5x^4 - 3x^{\frac{2}{3}} + 4$ then
    $$f'(x) = 5 \times 4x^{4-1} - 3 \times \frac{2}{3}x^{\frac{2}{3}-1} + 0$$
    $$f'(x) = 20x^3 - 2x^{-\frac{1}{3}}$$
- **Products** and **quotients cannot** be differentiated in this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = (2x-3)(x^2-4)$ then expand to $f(x) = 2x^3 - 3x^2 - 8x + 12$ which **is** a **sum/difference** of powers of x and can be differentiated

Worked example

The function $f(x)$ is given by

$$f(x) = 2x^3 + \frac{4}{\sqrt{x}}, \text{ where } x > 0$$

Find the derivative of $f(x)$

Rewrite f(x) so every term is a power of x

$f(x) = 2x^3 + 4x^{-\frac{1}{2}}$

Differentiate by applying the formula

$f'(x) = 6x^2 - 2x^{-\frac{3}{2}}$

anx^{n-1}

take care with negatives

$-\frac{1}{2} - 1 = -\frac{3}{2}$

$\boxed{\therefore f'(x) = 6x^2 - 2x^{-\frac{3}{2}}}$

5.1.2 Applications of Differentiation

Finding Gradients

How do I find the gradient of a curve at a point?

- The **gradient of a curve** at a point is the **gradient of the tangent** to the curve at that point
- **Find the gradient** of a curve at a point by **substituting the value of x** at that point into the curve's **derivative function**
- For example, if $f(x) = x^2 + 3x - 4$
  - then $f'(x) = 2x + 3$
  - and the gradient of $y = f(x)$ when $x = 1$ is $f'(1) = 2(1) + 3 = 5$
  - and the gradient of $y = f(x)$ when $x = -2$ is $f'(-2) = 2(-2) + 3 = -1$
- Although your GDC won't find a derivative function for you, it is possible to use your **GDC** to **evaluate the derivative** of a function at a point, using $\frac{d}{dx}(\square)_{x=\square}$

Worked example

A function is defined by $f(x) = x^3 + 6x^2 + 5x - 12$.

(a) Find $f'(x)$.

Find $f'(x)$ by differentiating

$f'(x) = 3x^2 + 2 \times 6x^1 + 5x^0$

$\boxed{f'(x) = 3x^2 + 12x + 5}$

(b) Hence show that the gradient of $y = f(x)$ when $x = 1$ is 20.

Substitute $x = 1$ into $f'(x)$

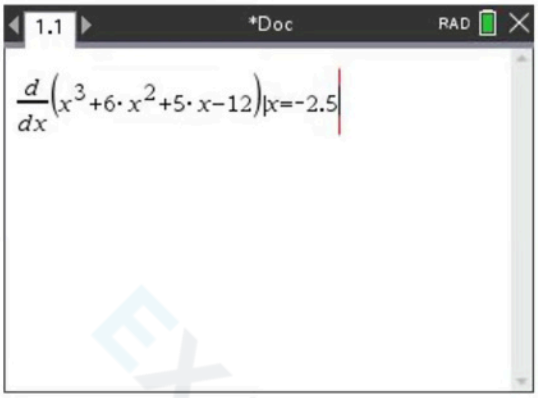
$f'(1) = 3(1)^2 + 12(1) + 5$

$= 3 + 12 + 5$

$\boxed{f'(1) = 20}$

(c) Find the gradient of $y = f(x)$ when $x = -2.5$.

Use the GDC to evaluate the derivative of $f(x)$ at $x = -2.5$

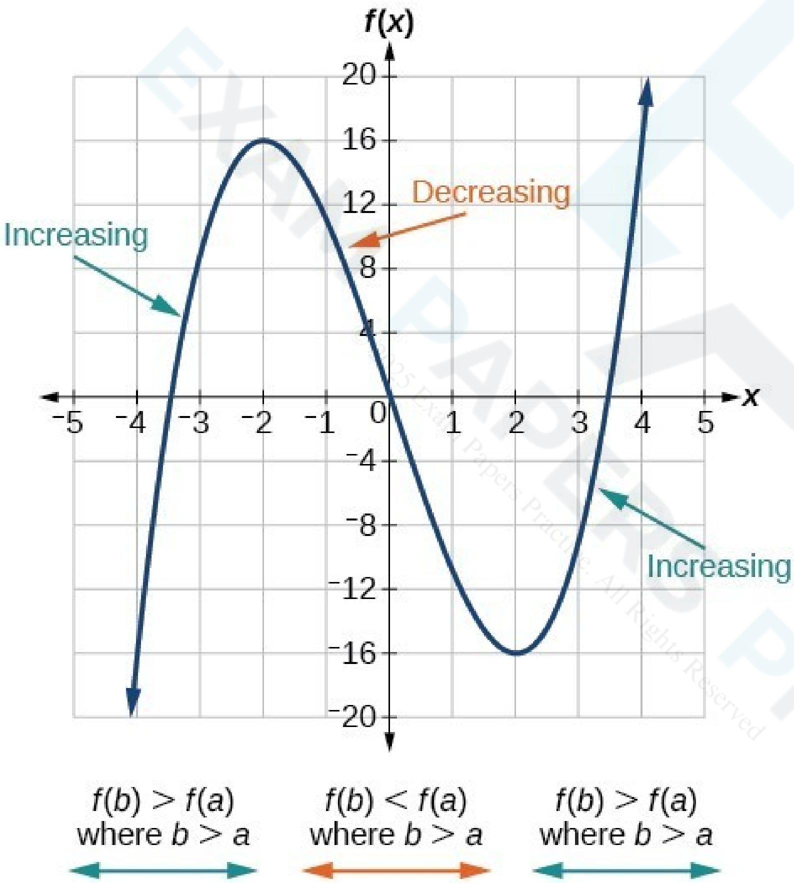


$\boxed{f'(-2.5) = -6.25}$

Increasing & Decreasing Functions

What are increasing and decreasing functions?

- A function, $f(x)$, is **increasing** if $\boldsymbol{f'(x) > 0}$
  - This means the **value** of the **function** ('output') **increases** as x **increases**
- A function, $f(x)$, is **decreasing** if $\boldsymbol{f'(x) < 0}$
  - This means the **value** of the **function** ('output') **decreases** as x **increases**
- A function, $f(x)$, is **stationary** if $\boldsymbol{f'(x) = 0}$



How do I find where functions are increasing, decreasing or stationary?

- To identify the **intervals** on which a function is increasing or decreasing

  STEP 1
  Find the derivative **f'(*x*)**

  STEP 2
  Solve the inequalities

$\boldsymbol{f'(x) > 0}$ (for increasing intervals) and/or
$\boldsymbol{f'(x) < 0}$ (for decreasing intervals)

- Most functions are a combination of **increasing, decreasing** and **stationary**
  - a range of values of x (**interval**) is given where a function satisfies each condition
  - e.g. The function $f(x) = x^2$ has **derivative** $f'(x) = 2x$ so
    - $f(x)$ is **decreasing** for $x < 0$
    - $f(x)$ is **stationary** at $x = 0$
    - $f(x)$ is **increasing** for $x > 0$

Worked example

$f(x) = x^2 - x - 2$

a) Determine whether $f(x)$ is increasing or decreasing at the points where $x = 0$ and $x = 3$.

Differentiate

$f'(x) = 2x - 1$

At $x = 0$, $f'(0) = 2 \times 0 - 1 = -1 < 0$ $\therefore$ decreasing

At $x = 3$, $f'(3) = 2 \times 3 - 1 = 6 > 0$ $\therefore$ increasing

$\therefore$ At $x = 0$, $f(x)$ is decreasing
At $x = 3$, $f(x)$ is increasing

b) Find the values of x for which $f(x)$ is an increasing function.

$f(x)$ is increasing when $f'(x) > 0$

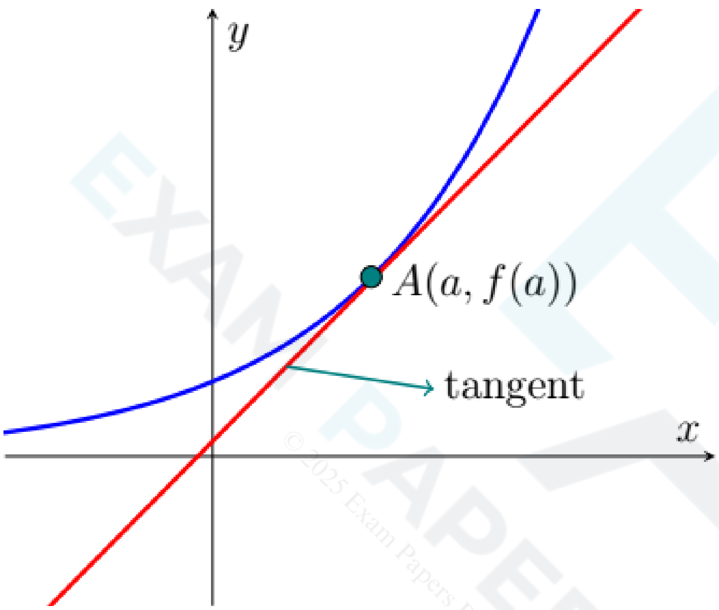
$f'(x) > 0$
$2x - 1 > 0$
$x > \frac{1}{2}$

$\therefore$ $f(x)$ is increasing for $x > \frac{1}{2}$

Tangents & Normals

What is a tangent?

- At any point on the graph of a (non-linear) **function**, the **tangent** is the straight line that touches the graph **at only that point**
- Its **gradient** is given by the **derivative function**

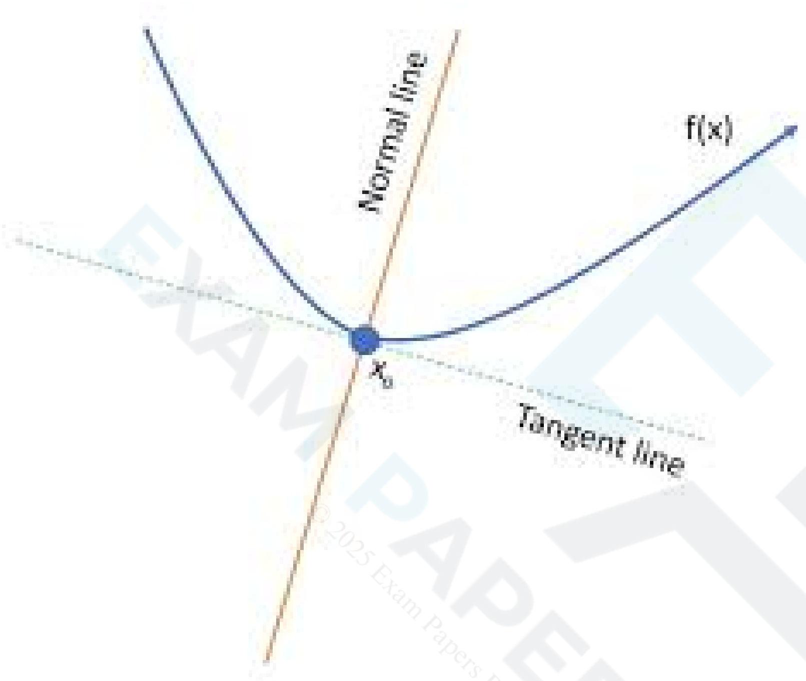


How do I find the equation of a tangent?

- To find the **equation of a straight line**, a **point** and the **gradient** are needed
- The **gradient**, m, of the **tangent** to the function $y = f(x)$ at (x_1, y_1) is $\boldsymbol{f'(x_1)}$
- Therefore find the **equation** of the **tangent** to the function $y = f(x)$ at the point (x_1, y_1) by substituting the gradient, $f'(x_1)$, and point (x_1, y_1) into $y - y_1 = m(x - x_1)$, giving:
  - $y - y_1 = \boldsymbol{f'(x_1)}(x - x_1)$
- (You could also substitute into $y = mx + c$ but it is usually quicker to substitute into $y - y_1 = m(x - x_1)$)

What is a normal?

- At any point on the graph of a (non-linear) function, the **normal** is the straight line that passes through that point and is **perpendicular** to the **tangent**



How do I find the equation of a normal?

- The **gradient** of the **normal** to the function $y = f(x)$ at (x_1, y_1) is $\frac{-1}{f'(x_1)}$
- Therefore find the **equation** of the **normal** to the function $y = f(x)$ at the point (x_1, y_1) by using

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

Worked example

The function $f(x)$ is defined by

$$f(x) = 2x^4 + \frac{3}{x^2} \qquad x \neq 0$$

a) Find an equation for the tangent to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $y = mx + c$.

First find $f'(x)$ by differentiating

$f(x) = 2x^4 + 3x^{-2}$ Rewrite as powers of x

$f'(x) = 8x^3 - 6x^{-3}$

For a tangent, "$y - y_1 = f'(a)(x - x_1)$"

At $x = 1$, $y = 2(1)^4 + \frac{3}{(1)^2} = 5$

$f'(1) = 8(1)^3 - \frac{6}{(1)^3} = 2$

$\therefore$ $y - 5 = 2(x - 1)$

$\boxed{\text{Tangent at } x = 1 \text{, is } y = 2x + 3}$

b) Find an equation for the normal at the point where $x = 1$, giving your answer in the form $ax + by + d = 0$, where a, b and d are integers.

For a normal, "$y - y_1 = \frac{-1}{f'(a)}(x - x_1)$"

Using results from part (a):

$y - 5 = \frac{-1}{2}(x - 1)$

$y = -\frac{1}{2}x + \frac{11}{2}$

$2y = -x + 11$

$\boxed{\therefore \text{Equation of normal is } x + 2y - 11 = 0}$

Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero i.e. $f'(x) = 0$
- A **local minimum** point, $(x,\ f(x))$ will be the **lowest** value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x,\ f(x))$ will be the **greatest** value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions tend to infinity for large values of x (and/or minus infinity for large negative values of x)
- The **nature** of a stationary point refers to whether it is a local **minimum** or local **maximum** point

How do I find the coordinates and nature of stationary points?

- The instructions below describe how to find **local minimum** and **maximum points** using a **GDC** on the graph of the function $y = f(x)$.

STEP 1
Plot the graph of $y = f(x)$
Sketch the graph as part of the solution

STEP 2
Use the options from the graphing screen to "solve for minimum"
The GDC will display the x and y coordinates of the first minimum point
Scroll onwards to see there are anymore minimum points
Note down the coordinates and the type of stationary point

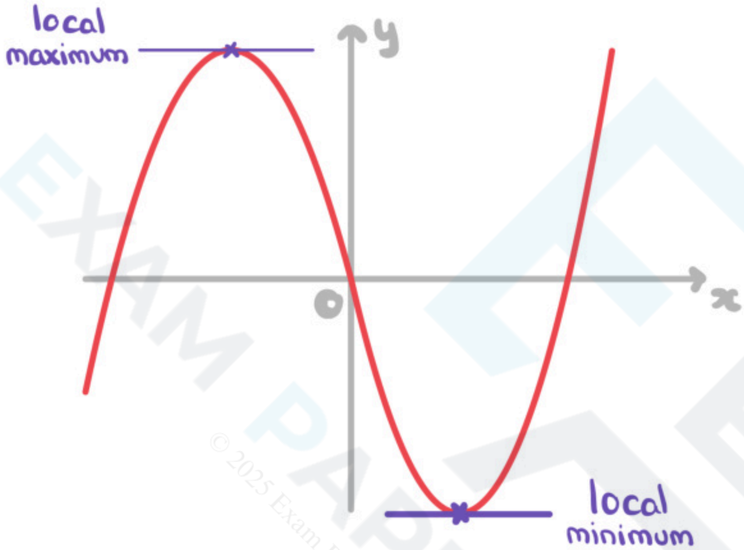
STEP 3
Repeat **STEP 2** but use "solve for maximum" on your GDC

- In **STEP 2** the **nature** of the stationary point should be easy to tell from the graph
  - a local **minimum** changes the function from **decreasing** to **increasing**
    - the gradient changes from **negative** to **positive**
  - a local **maximum** changes the function from **increasing** to **decreasing**
    - the gradient changes from **positive** to **negative**

Worked example

Find the stationary points of $f(x) = x(x^2 - 27)$, and state their nature.

Plot the graph of $y = x(x^2-27)$ on GDC and sketch here.



$\therefore$ Stationary points are

(3, -54) LOCAL MINIMUM POINT
(-3, 54) LOCAL MAXIMUM POINT

5.1.3 Modelling with Differentiation

Modelling with Differentiation

What can be modelled with differentiation?

- Recall that **differentiation** is about the **rate** of **change** of a function and provides a way of finding **minimum** and **maximum** values of a function
- Anything that involves **maximising** or **minimising** a quantity can be modelled using differentiation; for example
  - **minimising** the cost of raw materials in manufacturing a product
  - the **maximum** height a football could reach when kicked
- These are called **optimisation** problems

What modelling assumptions are used in optimisation problems?

- The quantity being **optimised** needs to be dependent on a **single** variable
  - If other variables are initially involved, **constraints** or **assumptions** about them will need to be made; for example
    - minimising the cost of the **main** raw material – timber in manufacturing furniture say – the cost of screws, glue, varnish, etc can be fixed or considered **negligible**
  - Other **modelling assumptions** may have to be made too; for example
    - ignoring air resistance and wind when modelling the path of a kicked football

How do I solve optimisation problems?

- In optimisation problems, letters other than x, y and f are often used including capital letters
  - V is often used for volume, S for surface area
  - r for radius if a circle, cylinder or sphere is involved
- **Derivatives** can still be found but be clear about which variable is independent (x) and which is dependent (y)
  - a GDC may always use x and y but ensure you use the correct variable throughout your working and final answer
- Problems often start by **linking two connected** quantities together – for example **volume** and **surface area**
  - where more than one variable is involved, **constraints** will be given such that the quantity of interest can be rewritten in terms of a **single** variable
- Once the quantity of interest is written as a function of a **single** variable, **differentiation** can be used to **maximise** or **minimise** the quantity as required

STEP 1

Rewrite the quantity to be optimised as a single variable, using any constraints given in the question

STEP 2

Use your GDC to find the (local) maximum or minimum points as required

Plot the graph of the function and use the graphing features of the GDC to “solve for minimum/maximum” as required

STEP 3

Note down the solution from your GDC and interpret the answer(s) in the context of the question

Worked example

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.

The total area of the bed is to be $100\pi \text{ m}^2$.

a) Show that the perimeter of the bed is given by the formula

$$P = \pi\left(r + \frac{100}{r}\right)$$

The width of the rectangle is $2r$ m and its length L m

The AREA of the bed, 100π m^2 is given by

$$\underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} + \underbrace{2rL}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}\pi r^2}_{\text{semi-circle}} = 100\pi$$

total area (this is the constraint)

$$\therefore \pi r^2 + 2rL = 100\pi$$

$$2rL = 100\pi - \pi r^2$$

$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

Write L in terms of r

The PERIMETER of the bed is

$$P = \underbrace{\pi r + \pi r}_{\text{semi-circular arcs}} + \underbrace{2L}_{\text{two straight lengths}}$$

Use L from the area constraint to write P in terms of r only

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$

$$P = \pi r + \frac{100\pi}{r}$$

$$\boxed{\therefore P = \pi\left(r + \frac{100}{r}\right)}$$

b) Find $\frac{\mathrm{d}P}{\mathrm{d}r}$.

Rewrite P as powers of r

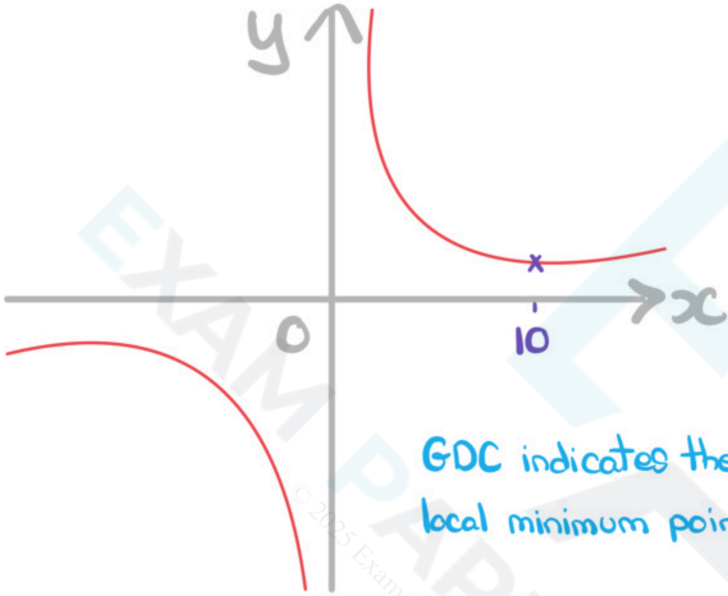
$$P = \pi(r + 100r^{-1})$$

$$\frac{dP}{dr} = \pi(1 - 100r^{-2})$$

$$\boxed{\therefore \frac{dP}{dr} = \pi\left(1 - \frac{100}{r^2}\right)}$$

c) Find the value of r that minimises the perimeter.

Use GDC to plot $y = \pi\left(x + \frac{100}{x}\right)$ and sketch the result



GDC indicates the ONLY local minimum point is at $x = 10$

$\therefore$ **The value of r that minimises the perimeter is $r = 10$**

d) Hence find the minimum perimeter.

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The minimum perimeter will be the y-coordinate of the local minimum point found in part (c)

From GDC. $y = 62.831\,853...$ (when $x = 10$)

$\therefore$ Minimum perimeter is 62.8 m (3 s.f.)