Exam Papers Practice

5.1 Differentiation

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Course	DP IB Maths
Section	5. Calculus
Topic	5.1 Differentiation
Difficulty	Medium

To be used by all students preparing for DP IB Maths AI SL
Students of other boards may also find this useful

Question 1

a) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$y = \frac{3}{2}x^2 - 15x + 2$

Apply formula

$\boxed{\frac{dy}{dx} = 3x - 15}$

b)i) Set $\frac{dy}{dx} = -3$ and solve for x.

$3x - 15 = -3$ add 15 then divide by 3

$x = 4$

Sub $x = 4$ into y.

$y = \frac{3}{2}(4)^2 - 15(4) + 2$

$y = -34$

$\boxed{\therefore A(4, -34)}$

ii) Sub A and $m = -3$ into $y - y_1 = m(x - x_1)$.

$y - (-34) = -3(x - 4)$ expand and rearrange

$\boxed{y = -3x - 22}$

Question 2

a) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$f(x) = 3x^7 - 12x$

Apply formula

$\boxed{f'(x) = 21x^6 - 12}$

b) Sub $x = 0$ into $f'(x)$.

$f'(0) = 21(0)^6 - 12.$

$\boxed{f'(0) = -12}$

c) The tangent and normal are perpendicular.

if $m_{normal} = 4$ then $m_{tangent} = -\frac{1}{4}$

Set $f'(x) = -\frac{1}{4}$ and solve for x.

$21x^6 - 12 = -\frac{1}{4}$

add 12, divide by 21 then $\sqrt[6]{}$.

$x = \pm \sqrt[6]{\frac{11.75}{21}} = \pm 0.9077...$

$x = \pm 0.908$ (3sf)

$\therefore y = \pm 9.3694... = \pm 9.37$ (3sf)

$\boxed{(0.908, -9.37) \text{ and } (-0.908, 9.37)}$

Question 3

a) Find $\frac{dy}{dx}$

$y = 4 - \frac{4}{x} = 4 - 4x^{-1}$

$\frac{dy}{dx} = \frac{4}{x^2} = 4x^{-2}$

Sub $x = 2$ into $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{4}{(2)^2} = \frac{4}{4} = 1$ $\therefore m = 1$

Sub $x = 2$ into y.

$y = 4 - \frac{4}{(2)} = 2$ $\therefore$ point $(2, 2)$

Sub m and the point into $y - y_1 = m(x - x_1)$.

$y - 2 = 1(x - 2)$

$\boxed{y = x}$

b) Set $\frac{dy}{dx} = 16$ and solve for x.

$\frac{4}{x^2} = 16$

reciprocate and multiply by 4

$x^2 = \frac{1}{4}$

$\sqrt{\ }$

$x = \pm\frac{1}{2}$

$\therefore y = -4$ and 12.

$\boxed{\left(\frac{1}{2}, -4\right) \text{ and } \left(-\frac{1}{2}, 12\right)}$

Question 4

a)i) Sub $x = 2$ into $f(x)$.

$$f(2) = \frac{4}{(2)} + \frac{2(2)^4}{5} - \frac{2}{5}$$

$$\boxed{f(2) = 8}$$

ii) Find $f'(x)$

$$f(x) = 4x^{-1} + \frac{2}{5}x^4 - \frac{2}{5}$$

$$f'(x) = -4x^{-2} + \frac{8}{5}x^3$$

Sub $x = 2$ into $f'(x)$.

$$f'(2) = -4(2)^{-2} + \frac{8}{5}(2)^3$$

$$\boxed{f'(2) = 11.8}$$

b) point $(2, 8)$ $\quad m = 11.8$

Sub m and the point into $y - y_1 = m(x - x_1)$.

$y - 8 = 11.8(x - 2)$

$y = 11.8x - 15.6$

expand and rearrange

$$\boxed{\text{The equation of } l \text{ is } y = 11.8x - 15.6.}$$

c) Graph $f(x)$ and l on your GDC and find their intersection.

$\boxed{A(-0.222, -18.2)}$

Question 5

a) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$f(x) = x^2 - bx + c$

Apply formula

$\boxed{f'(x) = 2x - b}$

b) Tangent equation at $x = 2$ is $y = x - 1$.

$\therefore f'(2) = 1$

$2(2) - b = 1$

$\boxed{b = 3}$

c) Sub $x = 2$ into $y = x - 1$.

$y = 2 - 1$

$y = 1$

$\therefore f(x)$ passes through $(2, 1)$.

$f(2) = 1$

$(2)^2 - 3(2) + c = 1$

$$\boxed{\begin{array}{l} c = 3 \\ f(x) = x^2 - 3x + 3 \end{array}}$$

Question 6

a) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$y = \frac{1}{35}x^5 - \frac{3}{4}x^3 + 6x$

Apply formula

$$\boxed{\frac{dy}{dx} = \frac{1}{7}x^4 - \frac{9}{4}x^2 + 6}$$

b) Graph y on your GDC and find its maximums.

Note that A has a negative x and y coordinate and B has a positive x and y coordinate.

i) $\boxed{A(-3.51, -3.85)}$

ii) $\boxed{B(1.84, 6.97)}$

c) The tangent and normal line are perpendicular.

$m_{normal} = -\frac{1}{10}$ $\quad \therefore m_{tangent} = 10$

Set $\frac{dy}{dx} = 10$ and solve for x.

$\frac{1}{7}x^4 - \frac{9}{4}x^2 + 6 = 10$

$x = \pm 4.1668... = \pm 4.17$ (3sf)

x-coordinate for R is 4.17 and the x-coordinate for S is -4.17.

Question 7

a) $x = 0$ when no shoes are produced.

$C(0) = 1225 + 11(0) - 0.009(0)^2 - 0.0001(0)^3$

$\boxed{C(0) = 1225 \text{ USD}}$

b) Derivative of x^n formula (in formula booklet)

$$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$$

$$C(x) = 1225 + 11x - 0.009x^2 - 0.0001x^3$$

Apply formula

$$\boxed{C'(x) = 11 - 0.018x - 0.0003x^2}$$

c) i) Sub $x = 40$ into $C'(x)$

$$C'(40) = 11 - 0.018(40) - 0.0003(40)^2$$

$$\boxed{C'(40) = 9.80 \text{ USD}}$$

ii) Sub $x = 90$ into $C'(x)$

$$C'(90) = 11 - 0.018(90) - 0.0003(90)^2$$

$$\boxed{C'(90) = 6.95 \text{ USD}}$$

d) Optimum level of production is when

$$R'(x) = C'(x)$$

$$4.5 = 11 - 0.018x - 0.0003x^2$$

Solve for x on your GDC

$$x = 120.222\ldots$$

120 pairs of running shoes.

Question 8

a) Sub $t = 60$ into $h(t)$.

$$h(60) = -\frac{1}{24}(60)^2 + 3(60) + 12$$

$$\boxed{h(60) = 42\text{ m}}$$

b) Derivative of x^n formula (in formula booklet)

$$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$$

$$h(t) = -\frac{1}{24}t^2 + 3t + 12$$

$$\boxed{h'(t) = -\frac{1}{12}t + 3}$$

c) Set $h'(t) = 0$ and solve for t.

$$-\frac{1}{12}t + 3 = 0$$

$$\boxed{t = 36\text{ s}}$$

Sub $t = 36$ into $h(t)$.

$$h(36) = -\frac{1}{24}(36)^2 + 3(36) + 12$$

$$\boxed{h(36) = 66\text{ m}}$$

Question 9

a) Sub $x = 0$ into $C(x)$.

$C(0) = 6(0)^3 - 10(0)^2 + 10(0) + 4$

$C(0) = 4$

$\boxed{400 \text{ AUD}}$

b) $P(x) = R(x) - C(x)$

$P(x) = 42x - (6x^3 - 10x^2 + 10x + 4)$

$\boxed{P(x) = -6x^3 + 10x^2 + 32x - 4}$

c) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$P(x) = -6x^3 + 10x^2 + 32x - 4$

$\boxed{P'(x) = -18x^2 + 20x + 32}$

d) Set $P'(x) = 0$ and solve for x.

$-18x^2 + 20x + 32 = 0$

$x = 2$ $\quad$ ~~$x = -0.889$~~

Reject as $P(x) > 0$ and $x \geq 0$

Sub $x = 2$ into $P(x)$.

$P(2) = -6(2)^3 + 10(2)^2 + 32(2) - 4$

$P(2) = 52$

∴ The profit maximising production level is 200 bats and the expected profit is 5200 AUD.

Question 10

a) Volume of a cuboid formula.

$V = lwh$ (in formula booklet)

where l is the length, w is the width and h is the height.

$l = 55 - 2x$ $\quad$ $w = 28 - 2x$ $\quad$ $h = x$

Sub l, w and h into formula.

$V = (55-2x)(28-2x)x$

expand and rearrange

$\boxed{V = 4x^3 - 166x^2 + 1540x}$

b) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$V = 4x^3 - 166x^2 + 1540x$

$$\boxed{\frac{dV}{dx} = 12x^2 - 332x + 1540}$$

c)i) Set $\frac{dV}{dx} = 0$ and solve for x.

$12x^2 - 332x + 1540 = 0$

$x = 5.8943...$ $\quad \cancel{x = 21.772...}$
Reject as $l > 0$

$$\boxed{x = 5.89\text{cm} \quad (3sf)}$$

ii) Sub $x = 5.8943...$ into V.

$V = 4(5.8943)^3 - 166(5.8943)^2 + 1540(5.8943)$

$V = 4129.059...$
$= 4130 \quad (3sf)$

$$\boxed{V = 4.13 \times 10^3 \text{ cm}^3}$$