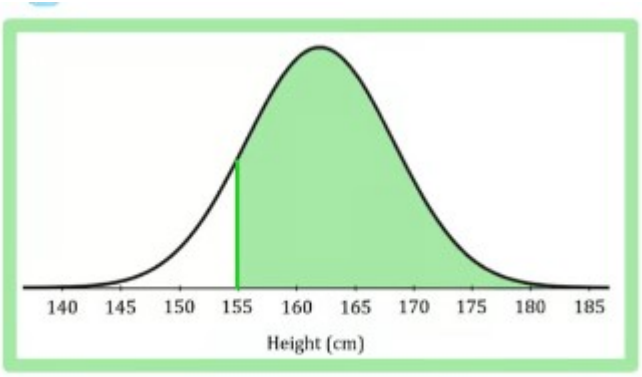
4.6 Normal Distributions

Mark Schemes

Course	DP IB Maths
Section	4. Statistics & Probability
Topic	4.6 Normal Distributions
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1



b)(i) Using the normal distribution on your calculator.

Lower = 155

Upper = 999...

$P(X > 155) = 0.866739...$

$P(X > 155) = 0.8667$ (4dp)

(ii) The normal distribution is symmetrical about its mean.

$P(X > \mu + k) = P(\mu < \mu - k)$

$P(X > 169) = P(X < 155)$

$= 1 - P(X > 155)$

$= 1 - 0.866739...$

$= 0.133260...$

$P(X > 169) = 0.1333$ (4dp)

c)(i) 68% of data lies between $\mu \pm \sigma$

$\mu + \sigma = 162 + 6.3 = 168.3$

$\mu - \sigma = 162 - 6.3 = 155.7$

68% of heights lie in range 155.7cm to 168.3cm

95 % of data lies between $\mu \pm 2\sigma$

$\mu + 2\sigma = 162 + 2(6.3) = 174.6$

$\mu - 2\sigma = 162 - 2(6.3) = 149.4$

95% of heights lie in range 149.4 cm to 174.6 cm

99.7 % of data lies between $\mu \pm 3\sigma$

$\mu + 3\sigma = 162 + 3(6.3) = 180.9$

$\mu - 3\sigma = 162 - 3(6.3) = 143.1$

99.7% of heights lie in range 143.1 cm to 180.9 cm

Question 2

a) For a normal distribution $P(X<k) = P(X \leq k)$

For probabilities use 4dp or 3sf (whichever is more accurate)

(i) Lower = -999...
Upper = 20

$P(X<20) = 0.226627...$

$P(X<20) = 0.2266$ (4dp)

(ii) Lower = 29
Upper = 999...

$P(X \geq 29) = 0.066807...$

$P(X \geq 29) = 0.0668$ (4dp)

(iii) Lower = 20
Upper = 29

$P(20 \leq X < 29) = 0.706565...$

$P(20 \leq X < 29) = 0.7066$ (4dp)

b) (i) Lower = -999...
Upper = 90

$P(Y \leqslant 90) = 0.252492...$

$P(Y \leqslant 90) = 0.2525$ (4dp)

(ii) Lower = 140
Upper = 999...

$P(Y > 140) = 0.003830...$

$P(Y > 140) = 0.00383$ (3sf)

(iii) $85 \leqslant Y \leqslant 115$

Lower = 85
Upper = 115

$P(85 \leqslant Y \leqslant 115) = 0.682689...$

$P(85 \leqslant Y \leqslant 115) = 0.6827$ (4dp)

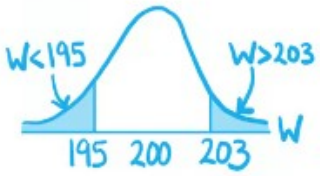
Question 3

(a) $W \sim N(200, 1.75^2)$

$\mu = 200$

$\sigma^2 = 1.75^2$

$\sigma = 1.75$



(i) Lower = -999...

Upper = 195

$P(W < 195) = 0.002137...$

$P(W < 195) = 0.00214$ (3sf)

(ii) Lower = 203

Upper = 999...

$P(W > 203) = 0.043238...$

$P(W > 203) = 0.0432$ (4 dp)

b) Let X be the number of chocolate bars in the sample that have a weight of at least 195g then

$X \sim B(12, p)$

$p = P(W \geq 195)$

$= 1 - P(W < 195)$

$= 1 - 0.002137...$ ← Use full answer to avoid rounding errors.

$= 0.997863...$

$\therefore X \sim B(12, 0.997863)$

If all of the sample exceed 195g then

$X = 12$

$P(X = 12) = (0.997863)^{12}$

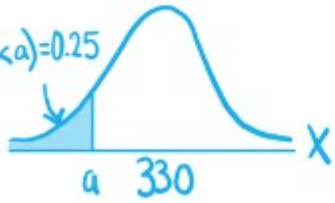
$= 0.974655...$

$P(X = 12) = 0.9747$ (4dp)

Question 4

(a) $X \sim N(330, 10^2)$ $\mu = 330$ $\sigma^2 = 10^2$ $\sigma = 10$ [

(i) $P(X<a)=0.25$

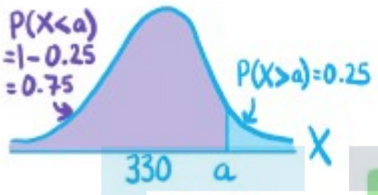


Inverse Normal
Area = 0.25

$a = 323.255...$

$a = 323.26$ (2dp)

(ii)

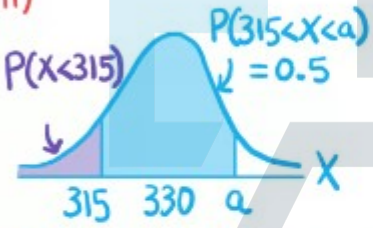


Inverse Normal
Area = 0.75 ← Area uses $P(X<a)$

$a = 336.744...$

$a = 336.74$ (2dp)

(iii)



$P(X<315) = 0.066807...$

$P(X<a) = P(X<315) + P(315<X<a)$

$P(X<a) = 0.066807... + 0.5$
$= 0.566807...$

Inverse Normal
Area = 0.566807

$a = 331.682...$

$a = 331.68$ (2dp)

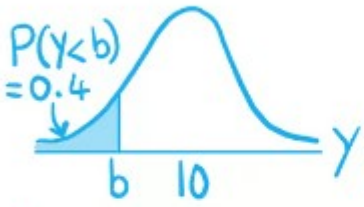
b) $Y \sim N(10, 10)$

$\mu = 10$

$\sigma^2 = 10$

$\sigma = \sqrt{10}$

(i)

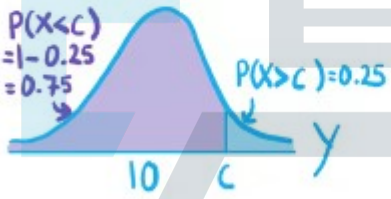


Inverse Normal
Area = 0.4

$b = 9.19884...$

$b = 9.20$ (2dp)

(ii)

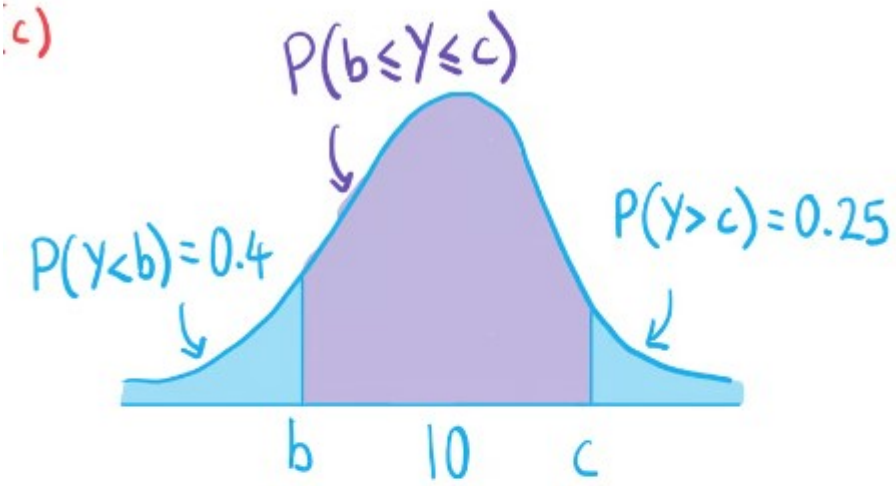


Inverse Normal
Area = 0.75

$C = 12.1329...$

$C = 12.13$ (2dp)

c)



The total area is 1

$$P(Y<b) + P(b \leq Y \leq c) + P(Y > c) = 1$$

$$P(b \leq Y \leq c) = 1 - P(Y<b) - P(Y > c)$$

$$= 1 - 0.4 - 0.25$$

$$= 0.35$$

Question 5

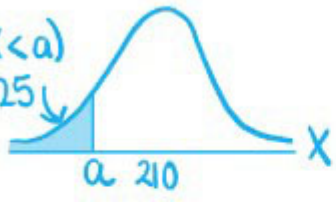
(a) $X \sim N(210\ 27.8^2)$

$\mu = 210$

$\sigma^2 = 27.8^2$

$\sigma = 27.8$

(i) $P(X < a) = 0.25$

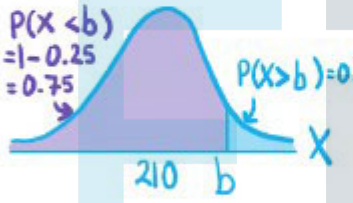


Inverse Normal

Area = 0.25

$a = 191.249...$

$a = 191$ (3sf)



Inverse Normal

Area = 0.75

$a = 228.750...$

$a = 229$ (3sf)

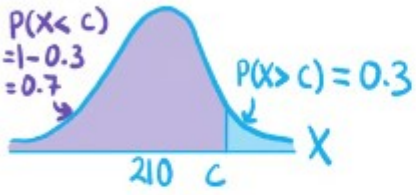
(ii) $P(X < LQ) = 0.25$ and $P(X > UQ) = 0.25$

$IQR = 228.750... - 191.249...$

$= 37.501...$

$IQR = 37.5$ (3sf)

b) In the top 30% means $X \geq c$ where $P(X \geq c) = 0.3$



Inverse Normal
Area = 0.7

$c = 224.578...$

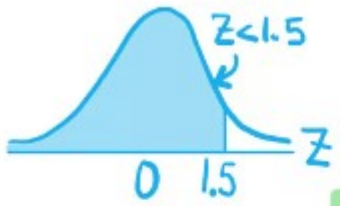
$231 > 224.578...$

Amelia is in the top 30% and will move on to the next stage of training.

Question 6

(a) $Z \sim N(0, 1^2)$ $\mu = 0$ $\sigma^2 = 1^2$ $\sigma = 1$

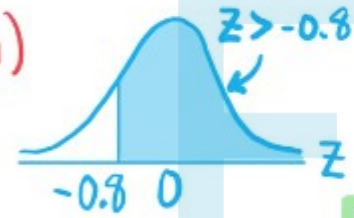
(i)



Lower = -999...
Upper = 1.5
$P(Z < 1.5) = 0.933192...$

$P(Z < 1.5) = 0.9332$ (4dp)

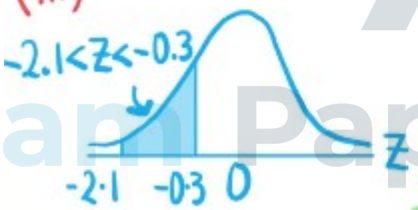
(ii)



Lower = -0.8
Upper = 999...
$P(Z > -0.8) = 0.788144...$

$P(Z > -0.8) = 0.7881$ (4dp)

(iii)



Lower = -2.1
Upper = -0.3
$P(-2.1 < Z < -0.3) = 0.364224...$

$P(-2.1 < z < -0.3) = 0.3642$ (4dp)

(b) $X \sim N(2, 0.1^2)$ $\mu = 2$ $\sigma^2 = 0.1^2$ $\sigma = 0.1$

$$Z = \frac{X - \mu}{\sigma}$$

$$z = \frac{x - 2}{0.1}$$

$$\therefore x = 2 + 0.1z$$

To find $x = a$ use $z = 1.5$

$a = 2 + 0.1(1.5) = 2.15$

$P(X < 2.15) = 0.9332$ (4dp)

To find $x = b$ use $z = -0.8$

$b = 2 + 0.1(-0.8) = 1.92$

$P(X > 1.92) = 0.7881$ (4dp)

To find $x = c$ use $z = -2.1$

$c = 2 + 0.1(-2.1) = 1.79$

To find $x = d$ use $z = -0.3$

$d = 2 + 0.1(-0.3) = 1.97$

$P(1.79 < X < 1.97) = 0.3642$ (4dp)

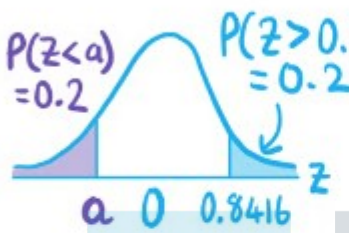
Question 7

a) (i) Using the percentage points table with $p = 0.2$

$z = 0.8416$

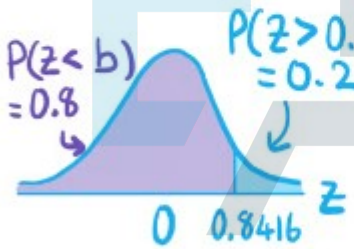
This could also be done using the calculator

(ii)



By symmetry of the curve

$a = -0.8416$



The total area under the curve is 1

$P(Z < b) = 0.8$

$\Rightarrow P(Z > b) = 0.2$

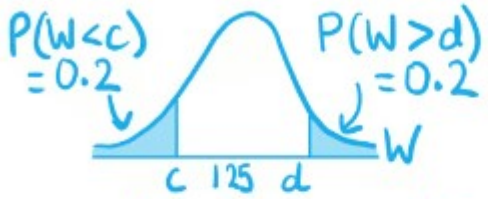
$b = 0.8416$

b) $\mu = 1.25$

$\sigma = 0.38$

$\sigma^2 = 0.38^2$

$W \sim N(1.25, 0.38^2)$



Using $Z = \frac{W-\mu}{\sigma}$ $z = \frac{w-1.25}{0.38}$ $\Rightarrow w = 1.25 + 0.38z$

When $w = c$, $z = -0.8416$

$c = 1.25 + 0.38(-0.8416) = 0.930192$ kg

When $w = d$, $z = 0.8416$

$d = 1.25 + 0.38(0.8416) = 1.569808$ kg

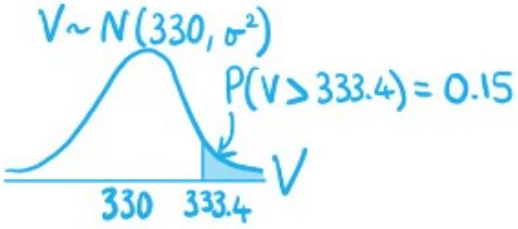
20% to 80% interpercentile range

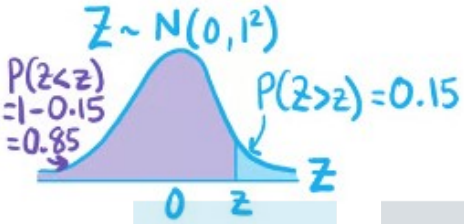
$0.93\text{kg} \leq W \leq 1.57\text{kg}$

Question 8

a) $\mu = 330$ $\quad V \sim N(330, \sigma^2)$

15% contain more than 333.4ml $\quad P(V > 333.4) = 0.15$





Inverse Normal
Area = 0.85
$z = 1.036432...$

Using $Z = \frac{V-\mu}{\sigma}$ $\quad Z = 1.036432$, $V = 333.4$

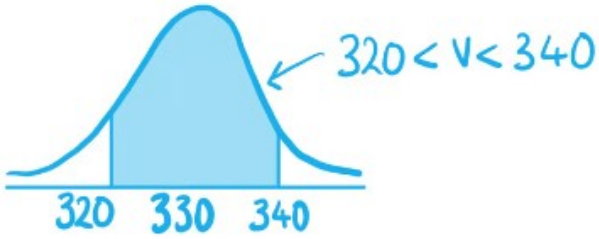
$$1.036432 = \frac{333.4 - 330}{\sigma}$$

$$\sigma = \frac{333.4 - 330}{1.036432}$$

$$\sigma = 3.28048...$$

$$\sigma = 3.28 \text{ (3sf)}$$

b) $V \sim N(330, 3.28^2)$



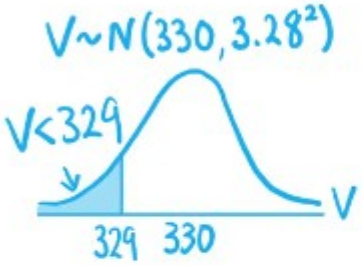
Lower = 320
Upper = 340

$P(320 < V < 340) = 0.997702...$

$$P(320 < V < 340) = 0.9977 \text{ (4dp)}$$

c) Let X be the number of cans that contain less than 329ml

$X \sim B(6, p)$



Lower = -999...
Upper = 329

$p = P(V < 329)$
$= 0.380250...$

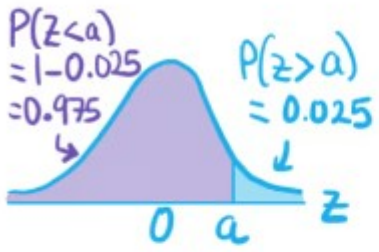
All of the cans contain less than 329ml
$X = 6$

$P(X=6) = (0.380250)^6$
$= 0.0030228...$

$P(X=6) = 0.00302$ (3sf)

Question 9

a) $Z \sim N(0, 1^2)$

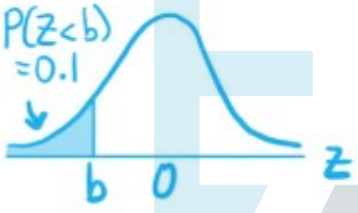


Inverse Normal
Area = 0.975

$a = 1.959964...$

$a = 1.9600$ (4dp)

Alternatively Percentage points table could be used.



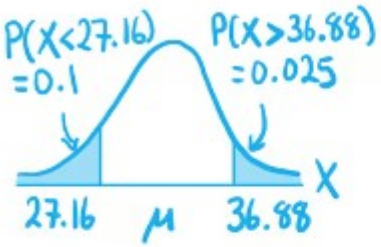
Inverse Normal
Area = 0.1

$b = -1.281551...$

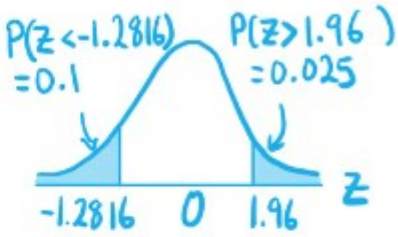
$b = -1.2816$ (4dp)

b)

$X \sim N(\mu, \sigma^2)$



$Z \sim N(0, 1^2)$



Using $Z = \frac{X-\mu}{\sigma}$

$z = -1.2816$, $x = 27.16$

$-1.2816 = \frac{27.16-\mu}{\sigma}$

$\mu - 1.2816\sigma = 27.16$

$z = 1.96$, $x = 36.88$

$1.96 = \frac{36.88-\mu}{\sigma}$

$\mu + 1.96\sigma = 36.88$

c) Simultaneous equations

$\mu + 1.96\sigma = 36.88$ ①

$-(\mu - 1.2816\sigma = 27.16)$ ② Subtract equations to get rid of μ

$3.2416\sigma = 9.72$

$\sigma = \frac{9.72}{3.2416}$

$\sigma = 2.998519...$

Substitute σ back into ① to find μ

$\mu + 1.96(2.998519) = 36.88$ Use full answer for σ to avoid rounding errors.

$\mu = 36.88 - 1.96(2.998519)$

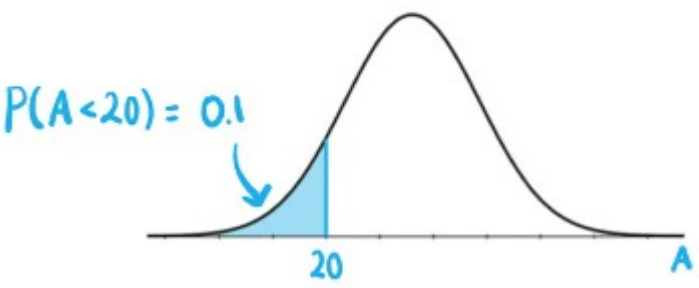
$\mu = 31.002902...$

$\mu = 31.00$ (2dp)

$\sigma = 3.00$ (2dp)

Question 10

$A \sim N(22.5, \sigma^2)$ and $Z \sim N(0, 1^2)$



(i) Using the inverse normal distribution with:

Area = 0.1 $\quad \mu = 0 \quad \sigma = 1$

$P(Z < z) = 0.1 \Rightarrow z = -1.2815516...$

Using $z = \frac{x - \mu}{\sigma}$

$-1.2815516... = \frac{20 - 22.5}{\sigma}$

$\sigma = \frac{20 - 22.5}{-1.2815516...}$

$\sigma = 1.950760...$

$\sigma = 1.95$ (3sf)

(ii) Use the unrounded value of $\sigma = 1.950760...$

Lower = $-999...$

Upper = 18

$P(A < 18) = 0.010533...$

$P(A < 18) = 0.0105$ (3sf)