DP IB Maths: AA HL

4.6 Normal Distribution

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4.6.1 The Normal Distribution

Properties of Normal Distribution

The binomial distribution is an example of a discrete probability distribution. The normal distribution is an example of a **continuous** probability distribution.

What is a continuous random variable?

- A continuous random variable (often abbreviated to CRV) is a random variable that can take **any value** within a range of infinite values
  - Continuous random variables **usually measure** something
  - For example, height, weight, time, etc

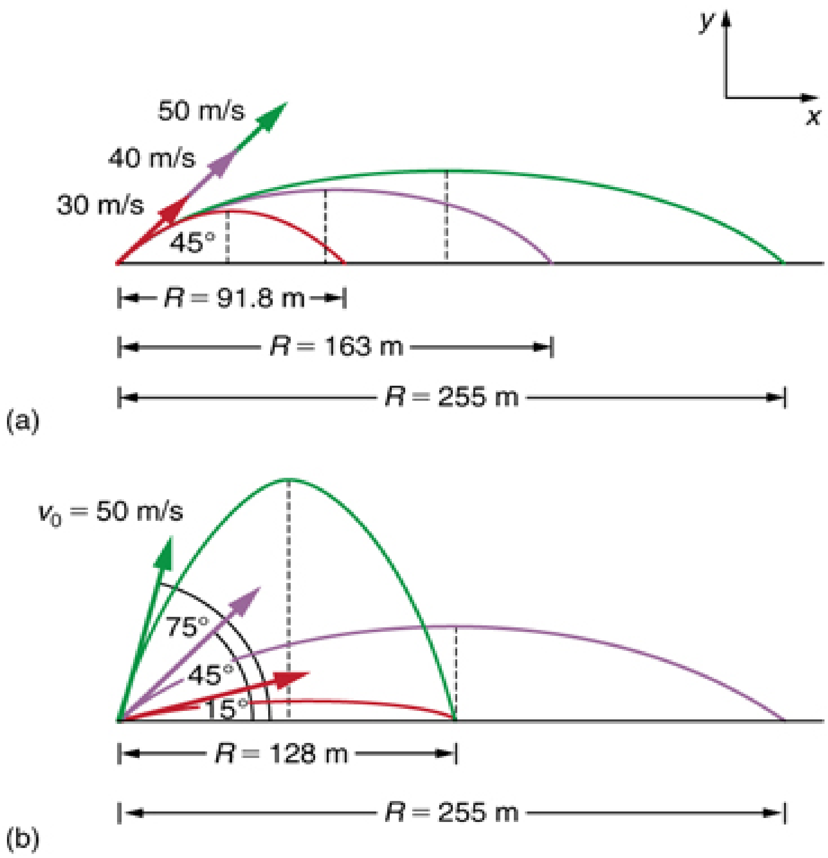
What is a continuous probability distribution?

- A continuous probability distribution is a probability distribution in which the random variable X is continuous
- The probability of X being a **particular value is always zero**
  - $\mathrm{P}(X = k) = 0$ for any value *k*
  - Instead we define the **probability density function** $\mathrm{f}(x)$ for a specific value
    - This is a function that describes the **relative likelihood** that the random variable would be close to that value
  - We talk about the **probability** of X being within a **certain range**
- A continuous probability distribution can be represented by a continuous graph (the values for X along the horizontal axis and probability **density** on the vertical axis)
- The **area under the graph** between the points $x = a$ and $x = b$ is equal to $\mathrm{P}(a \le X \le b)$
  - The **total area under the graph equals 1**
- As $\mathrm{P}(X = k) = 0$ for any value *k*, it does not matter if we use strict or weak inequalities
  - $\mathrm{P}(X \le k) = \mathrm{P}(X < k)$ for any value *k* when *X* is a **continuous random variable**

What is a normal distribution?

- A normal distribution is a **continuous probability distribution**
- The **continuous random variable** X can follow a normal distribution if:
  - The distribution is **symmetrical**
  - The distribution is **bell-shaped**
- If X follows a normal distribution then it is denoted $X \sim \mathrm{N}(\mu,\ \sigma^2)$
  - μ is the **mean**
  - σ^2 is the **variance**
  - σ is the **standard deviation**
- If the **mean** changes then the graph is **translated horizontally**

- If the **variance** increases then the graph is **widened horizontally** and **made shorter vertically** to maintain the same area
  - A **small variance** leads to a **tall** curve with a **narrow** centre
  - A **large variance** leads to a **short** curve with a **wide** centre



What are the important properties of a normal distribution?

- The **mean** is μ
- The **variance** is σ^2
  - If you need the **standard deviation** remember to square root this
- The normal distribution is symmetrical about $x = \mu$
  - Mean = Median = Mode = μ
- There are the results:
  - Approximately **two-thirds (68%)** of the data lies within **one standard deviation** of the mean ($\mu \pm \sigma$)
  - Approximately **95%** of the data lies within **two standard deviations** of the mean ($\mu \pm 2\sigma$)
  - Nearly **all of the data (99.7%)** lies within **three standard deviations** of the mean ($\mu \pm 3\sigma$)

Modelling with Normal Distribution

What can be modelled using a normal distribution?

- A lot of real-life continuous variables can be modelled by a normal distribution provided that the population is large enough and that the variable is **symmetrical** with **one mode**
- For a normal distribution X can take any real value, however values far from the mean (more than 4 standard deviations away from the mean) have a probability density of **practically zero**
  - This fact allows us to model variables that are not defined for all real values such as height and weight

What can not be modelled using a normal distribution?

- Variables which have **more than one mode** or **no mode**
  - For example: the number given by a random number generator
- Variables which are **not symmetrical**
  - For example: how long a human lives for

Worked example

The random variable S represents the speeds (mph) of a certain species of cheetahs when they run. The variable is modelled using $\mathrm{N}(40,\ 100)$.

a) Write down the mean and standard deviation of the running speeds of cheetahs.

$\mu = 40$ and $\sigma^2 = 100$

↑ Square root to get standard deviation

Mean $\mu = 40$
Standard deviation $\sigma = 10$

b) State two assumptions that have been made in order to use this model.

We assume that the distribution of the speeds is
- symmetrical
- bell-shaped

4.6.2 Calculations with Normal Distribution

Calculating Normal Probabilities

Throughout this section we will use the random variable $X \sim \mathrm{N}(\mu,\ \sigma^2)$. For *X* distributed normally, *X* can take any real number. Therefore any values mentioned in this section will be assumed to be real numbers.

How do I find probabilities using a normal distribution?

- The **area under a normal curve** between the points $x = a$ and $x = b$ is equal to the **probability** $\mathrm{P}(a < X < b)$
  - Remember for a normal distribution you do not need to worry about whether the inequality is strict (< or >) or weak (≤ or ≥)
    - $\mathrm{P}(a < X < b) = \mathrm{P}(a \le X \le b)$
- You will be **expected to use** distribution functions on your **GDC** to find the probabilities when working with a normal distribution

How do I calculate P(*X* = *x*): the probability of a single value for a normal distribution?

- The probability of a **single value** is **always zero** for a normal distribution
  - You can picture this as the area of a single line is zero
- $\mathrm{P}(X = x) = 0$
- Your GDC is likely to have a "**Normal Probability Density**" function
  - This is sometimes shortened to NPD, Normal PD or Normal Pdf
  - **IGNORE THIS FUNCTION** for this course!
  - This calculates the **probability density function** at a point **NOT the probability**

How do I calculate P(*a* < *X* < *b*): the probability of a range of values for a normal distribution?

- You need a **GDC** that can calculate **cumulative normal probabilities**
- You want to use the "**Normal Cumulative Distribution**" function
  - This is sometimes shortened to NCD, Normal CD or Normal Cdf
- You will need to enter:
  - The 'lower bound' - this is the value *a*
  - The 'upper bound' - this is the value *b*
  - The 'μ' value - this is the mean
  - The 'σ' value - this is the standard deviation
- **Check the order carefully** as some calculators ask for standard deviation before mean
  - Remember it is the standard deviation
    - so if you have the **variance** then **square root it**
- **Always sketch** a quick diagram to visualise which area you are looking for

How do I calculate P(*X* > *a*) or P(*X* < *b*) for a normal distribution?

- You will still use the "**Normal Cumulative Distribution"** function
- $P(X > a)$ can be estimated using an **upper bound that is sufficiently bigger** than the **mean**
  - Using a value that is more than 4 standard deviations **bigger than the mean** is quite accurate
  - Or an easier option is just to input lots of 9's for the upper bound (**99999999...** or **10^{99}**)
- $P(X < b)$ can be estimated using a **lower bound that is sufficiently smaller** than the **mean**
  - Using a value that is more than 4 standard deviations **smaller than the mean** is quite accurate
  - Or an easier option is just to input lots of 9's for the lower bound with a negative sign (**-99999999...** or **-10^{99}**)

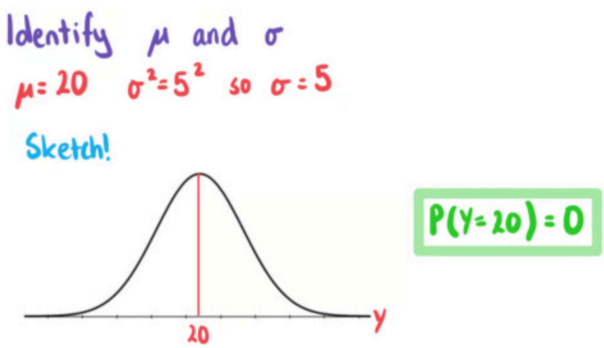
Are there any useful identities?

- $P(X < \mu) = P(X > \mu) = 0.5$
- As $P(X = a) = 0$ you can use:
  - $P(X < a) + P(X > a) = 1$
  - $P(X > a) = 1 - P(X < a)$
  - $P(a < X < b) = P(X < b) - P(X < a)$
- These are useful when:
  - The mean and/or standard deviation are unknown
  - You only have a diagram
  - You are working with the **inverse distribution**

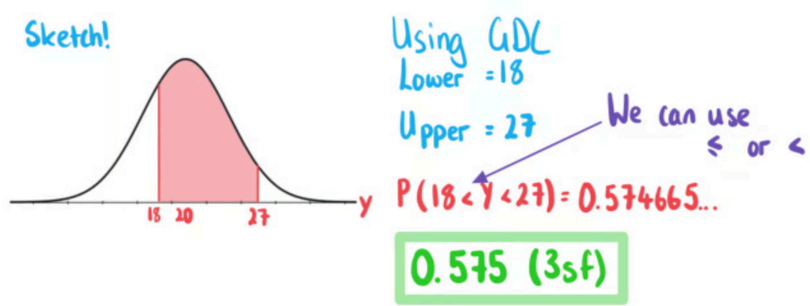
Worked example

The random variable $Y \sim N(20, 5^2)$. Calculate:

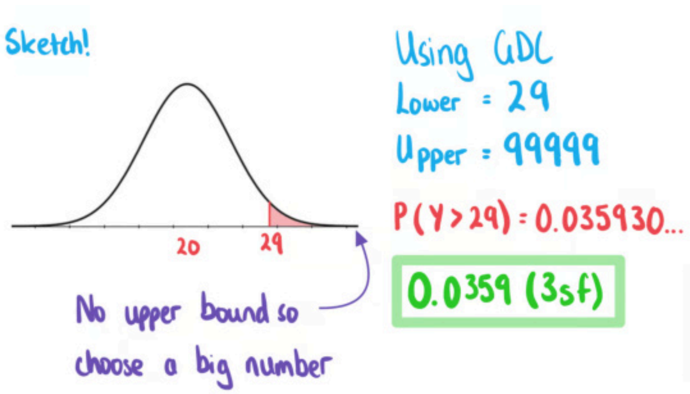
i) $P(Y = 20)$.



ii) $P(18 \leq Y < 27)$.



iii) $P(Y > 29)$



Inverse Normal Distribution

Given the value of P(*X* < *a*) how do I find the value of *a*?

- Your **GDC** will have a function called "**Inverse Normal Distribution**"
  - Some calculators call this InvN
- Given that $\mathrm{P}(X < a) = p$ you will need to enter:
  - The 'area' - this is the value *p*
    - Some calculators might ask for the 'tail' - this is the left tail as you know the area to the left of *a*
  - The 'μ' value - this is the mean
  - The 'σ' value - this is the standard deviation

Given the value of P(*X* > *a*) how do I find the value of *a*?

- If your calculator **does** have the **tail option** (left, right or centre) then you can use the "Inverse Normal Distribution" function straightaway by:
  - Selecting 'right' for the tail
  - Entering the area as '*p*'
- If your calculator **does not** have the **tail option** (left, right or centre) then:
  - Given $\mathrm{P}(X > a) = p$
  - Use $\mathrm{P}(X < a) = 1 - \mathrm{P}(X > a)$ to rewrite this as
    - $\mathrm{P}(X < a) = 1 - p$
  - Then use the **method for P(*X* < *a*)** to find *a*

Worked example

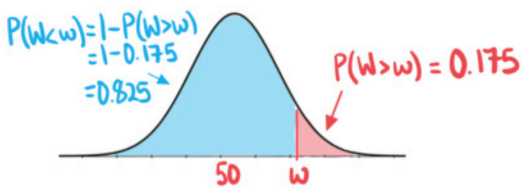
The random variable $W \sim \mathrm{N}(50, 36)$.

Find the value of w such that $\mathrm{P}(W > w) = 0.175$.

Identify μ and σ

$\mu = 50$ $\sigma^2 = 36$ so $\sigma = 6$

Sketch!



$\mathrm{P}(W > w)$ is less than 0.5 so w is bigger than the mean

Area from left is 0.825

Use Inverse Normal Distribution function on GDC

$w = 55.6075...$

$w = 55.6$ (3sf)

4.6.3 Standardisation of Normal Variables

Standard Normal Distribution

What is the standard normal distribution?

- The **standard normal distribution** is a normal distribution where the **mean is 0** and the **standard deviation is 1**
  - It is denoted by Z
  - $Z \sim \mathrm{N}(0,\ 1^2)$

Why is the standard normal distribution important?

- Any **normal distribution curve** can be transformed to the standard normal distribution curve by a **horizontal translation** and a **horizontal stretch**
- Therefore we have the relationship:
  - $Z = \dfrac{X - \mu}{\sigma}$
  - Where $X \sim \mathrm{N}(\mu,\ \sigma^2)$ and $Z \sim \mathrm{N}(0,\ 1^2)$
- Probabilities are related by:
  - $\mathrm{P}(a < X < b) = \mathrm{P}\left(\dfrac{a-\mu}{\sigma} < Z < \dfrac{b-\mu}{\sigma}\right)$
  - This will be useful when the mean or variance is unknown
- Some mathematicians use the function $\Phi(z)$ to represent $\mathrm{P}(Z < z)$

z-values

What are *z*-values (standardised values)?

- For a normal distribution $X \sim \mathrm{N}(\mu,\ \sigma^2)$ the *z*-value (standardised value) of an *x*-value tells you how many standard deviations it is away from the mean
  - If *z* = 1 then that means the *x*-value is 1 standard deviation bigger than the mean
  - If *z* = -1 then that means the *x*-value is 1 standard deviation smaller than the mean
- If the *x*-value is **more than the mean** then its corresponding *z*-value will be **positive**
- If the *x*-value is **less than the mean** then its corresponding *z*-value will be **negative**
- The *z*-value can be calculated using the formula:
  - $z = \dfrac{x - \mu}{\sigma}$
  - This is given in the **formula booklet**
- *z*-values can be used to compare values from different distributions

Finding Sigma and Mu

How do I find the mean (μ) or the standard deviation (σ) if one of them is unknown?

- If the **mean** or **standard deviation** of $X \sim \mathrm{N}(\mu,\ \sigma^2)$ is **unknown** then you will need to use the **standard normal distribution**
- You will need to use the formula
  - $z = \dfrac{x-\mu}{\sigma}$ or its rearranged form $x = \mu + \sigma z$
- You will be given a **probability for a specific value** of
  - $\mathrm{P}(X < x) = p$ or $\mathrm{P}(X > x) = p$
- To find the unknown parameter:
- **STEP 1**: **Sketch** the normal curve
  - Label the known value and the mean
- **STEP 2**: **Find** the ***z*-value** for the given value of ***x***
  - Use the **Inverse Normal Distribution** to find the value of z such that $\mathrm{P}(Z < z) = p$ or $\mathrm{P}(Z > z) = p$
  - Make sure the direction of the inequality for Z is consistent with the inequality for X
  - Try to **use lots of decimal places** for the *z*-value or **store your answer** to **avoid rounding errors**
    - You should use at least one extra decimal place within your working than your intended degree of accuracy for your answer
- **STEP 3**: **Substitute** the known values into $z = \dfrac{x-\mu}{\sigma}$ or $x = \mu + \sigma z$
  - You will be given and one of the parameters (*μ* or *σ*) in the question
  - You will have calculated *z* in STEP 2
- **STEP 4**: **Solve** the equation

How do I find the mean (μ) and the standard deviation (σ) if both of them are unknown?

- If **both** of them are **unknown** then you will be given two probabilities for two specific values of ***x***
- The process is the same as above
  - You will now be able to **calculate two *z* -values**
  - You can form **two equations** (rearranging to the form $x = \mu + \sigma z$ is helpful)
  - You now have to **solve the two equations simultaneously** (you can use your calculator to do this)
  - Be careful not to mix up which z-value goes with which value of *x*

Worked example

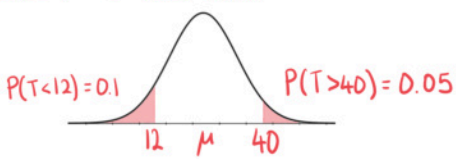
It is known that the times, in minutes, taken by students at a school to eat their lunch can be modelled using a normal distribution with mean μ minutes and standard deviation σ minutes.

Given that 10% of students at the school take less than 12 minutes to eat their lunch and 5% of the students take more than 40 minutes to eat their lunch, find the mean and standard deviation of the time taken by the students at the school.

Let $T \sim N(\mu, \sigma^2)$ be the time taken to eat lunch

STEP 1

Sketch the information



STEP 2

Find the corresponding z-values using inverse normal on GDC

$Z \sim N(0, 1^2)$

$P(Z < z_1) = 0.1 \Rightarrow z_1 = -1.2815...$

$P(Z > z_2) = 0.05 \Rightarrow P(Z < z_2) = 0.95 \Rightarrow z_2 = 1.6448...$

STEP 3

Form equations using $z = \frac{x - \mu}{\sigma}$ or $x = \mu + \sigma z$

$12 = \mu - (1.2815...)\sigma$

$40 = \mu + (1.6448...)\sigma$

STEP 4

Solve equations using GDC

$\mu = 24.26... \quad \sigma = 9.568...$

Mean = 24.3 mins (3sf)
Standard deviation = 9.57 mins (3sf)