DP IB Maths: AA HL

4.5 Binomial Distribution

Contents

4.5.1 The Binomial Distribution

Properties of Binomial Distribution

What is a binomial distribution?

- A binomial distribution is a **discrete probability distribution**
- A **discrete random variable** X follows a **binomial distribution** if it **counts the number of successes** when an experiment satisfies the following conditions:
  - There are a **fixed finite number of trials (*n*)**
  - The outcome of each trial is **independent** of the outcomes of the other trials
  - There are **exactly two outcomes** of each trial (**success or failure**)
  - The **probability of success is constant (*p*)**
- If X follows a binomial distribution then it is denoted $X \sim \mathrm{B}(n, p)$
  - *n* is the **number of trials**
  - *p* is the **probability of success**
- The **probability of failure is 1 – *p*** which is sometimes denoted as ***q***
- The formula for the probability of ***r* successful trials** is given by:
  - $\mathrm{P}(X = r) = {}^nC_r \times p^r(1-p)^{n-r}$ for $r = 0, 1, 2, \ldots, n$
    - ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$
  - You will be expected to use the distribution function on your **GDC to calculate probabilities** with the binomial distribution

What are the important properties of a binomial distribution?

- The **expected number (mean)** of successful trials is

$$\mathrm{E}(X) = np$$

  - You are given this in the **formula booklet**
- The **variance** of the number of successful trials is

$$\mathrm{Var}(X) = np(1-p)$$

  - You are given this in the **formula booklet**
  - Square root to get the **standard deviation**
- The distribution can be represented visually using a vertical line graph
  - If *p* is **close to 0** then the graph has a **tail to the right**
  - If *p* is **close to 1** then the graph has a **tail to the left**
  - If *p* is **close to 0.5** then the graph is **roughly symmetrical**
  - **If *p* = 0.5** then the graph is **symmetrical**

Modelling with Binomial Distribution

How do I set up a binomial model?

- **Identify** what a **trial** is in the scenario
  - For example: rolling a dice, flipping a coin, checking hair colour
- **Identify** what the **successful outcome** is in the scenario
  - For example: rolling a 6, landing on tails, having black hair
- **Identify** the **parameters**
  - *n* is the number of trials and *p* is the probability of success in each trial
- Make sure you **clearly state** what your **random variable** is
  - For example, let X be the number of students in a class of 30 with black hair

What can be modelled using a binomial distribution?

- Anything that satisfies the **four conditions**
- For example: let T be the number of times a fair coin lands on tails when flipped 20 times:
  - A trial is flipping a coin: There are 20 trials so ***n* = 20**
  - We can assume each coin flip does not affect subsequent coin flips: they are **independent**
  - A success is when the coin lands on tails: **Two outcomes** - tails or not tails (heads)
  - The coin is fair: The probability of tails is constant with ***p* = 0.5**
- Sometimes it might **seem like there are more than two outcomes**
  - For example: let Y be the number of yellow cars that are in a car park full of 100 cars
    - Although there are more than two possible colours of cars, here the trial is whether a car is yellow so there are two outcomes (yellow or not yellow)
    - Y would still need to fulfil the other conditions in order to follow a binomial distribution
- Sometimes a **sample may be taken from a population**
  - For example: 30% of people in a city have blue eyes, a sample of 30 people from the city is taken and X is the number of them with blue eyes
    - As long as the population is large and the sample is random then it can be assumed that each person has a 30% chance of having blue eyes

What can not be modelled using a binomial distribution?

- Anything where the number of trials is **not fixed** or is **infinite**
  - The number of emails received in an hour
  - The number of times a coin is flipped until it lands on heads
- Anything where the outcome of **one trial affects** the outcome of the **other trials**
  - The number of caramels that a person eats when they eat 5 sweets from a bag containing 6 caramels and 4 marshmallows
    - If you eat a caramel for your first sweet then there are less caramels left in the bag when you choose your second sweet
  - Anything where there are **more than two outcomes** of a trial
    - A person's shoe size
    - The number a dice lands on when rolled

- Anything where the **probability of success changes**
  - The number of times that a person can swim a length of a swimming pool in under a minute when swimming 50 lengths
    - The probability of swimming a lap in under a minute will decrease as the person gets tired
    - The probability is **not constant**

Worked example

It is known that 8% of a large population are immune to a particular virus. Mark takes a sample of 50 people from this population. Mark uses a binomial model for the number of people in his sample that are immune to the virus.

a) State the distribution that Mark uses.

A trial is checking if a person is immune to the virus

A success is if the person is immune.

Let X be the number of people in the sample immune to the virus

$X \sim B(50, 0.08)$

Number of people in sample (50)

Probability of being immune to the virus (0.08)

b) State two assumptions that Mark must make in order to use a binomial model.

Mark needs to assume that:

- each person in the population has an 8% chance of being immune
- the sample is random and the people are independent a person being immune does not affect the immunity of others

For example: If all 50 came from the same family then they would not be independent

c) Calculated the expected number of people in the sample that are immune to the virus.

Formula booklet

Binomial distribution $X \sim B(n, p)$	
Mean	$E(X) = np$

$E(X) = 50 \times 0.08$

4 people

4.5.2 Calculating Binomial Probabilities

Calculating Binomial Probabilities

Throughout this section we will use the random variable $X \sim \mathrm{B}(n, p)$. For binomial, the probability of X taking a non-integer or negative value is always zero. Therefore any values of X mentioned in this section will be assumed to be non-negative integers.

How do I calculate P(*X* = *x*): the probability of a single value for a binomial distribution?

- You should have a **GDC** that can calculate **binomial probabilities**
- You want to use the "**Binomial Probability Distribution**" function
  - This is sometimes shortened to BPD, Binomial PD or Binomial Pdf
- You will need to enter:
  - The '*x*' value - the value of *x* for which you want to find $\mathrm{P}(X = x)$
  - The '*n*' value - the **number of trials**
  - The '*p*' value - the **probability of success**
- Some calculators will give you the option of **listing the probabilities** for **multiple values of *x* at once**
- There is a formula that you can use but you are expected to be able to use the distribution function on your GDC
  - $\mathrm{P}(X = x) = {}^{n}\mathrm{C}_{x} \times p^{x}(1 - p)^{n - x}$
    - ${}^{n}\mathrm{C}_{x} = \dfrac{n!}{r!(n - r)!}$

How do I calculate P(*a* ≤ *X* ≤ *b*): the cumulative probabilities for a binomial distribution?

- You should have a **GDC** that can calculate **cumulative binomial probabilities**
  - Most calculators will find $\mathrm{P}(a \le X \le b)$
  - Some calculators can only find $\mathrm{P}(X \le b)$
    - The identities below will help in this case
- You should use the "**Binomial Cumulative Distribution**" function
  - This is sometimes shortened to BCD, Binomial CD or Binomial Cdf
- You will need to enter:
  - The lower value - this is the **value *a***
    - This can be zero in the case $\mathrm{P}(X \le b)$
  - The upper value - this is the **value *b***
    - This can be *n* in the case $\mathrm{P}(X \ge a)$
  - The '*n*' value - the **number of trials**
  - The '*p*' value - the **probability of success**

How do I find probabilities if my GDC only calculates P(*X* ≤ *x*)?

- To calculate $P(X \leq x)$ just enter x into the cumulative distribution function

- To calculate $P(X < x)$ use:
  - $P(X < x) = P(X \leq x - 1)$ which works when X is a binomial random variable
    - $P(X < 5) = P(X \leq 4)$

- To calculate $P(X > x)$ use:
  - $P(X > x) = 1 - P(X \leq x)$ which works for any random variable X
    - $P(X > 5) = 1 - P(X \leq 5)$

- To calculate $P(X \geq x)$ use:
  - $P(X \geq x) = 1 - P(X \leq x - 1)$ which works when X is a binomial random variable
    - $P(X \geq 5) = 1 - P(X \leq 4)$

- To calculate $P(a \leq X \leq b)$ use:
  - $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a - 1)$ which works when X is a binomial random variable
    - $P(5 \leq X \leq 9) = P(X \leq 9) - P(X \leq 4)$

What if an inequality does not have the equals sign (strict inequality)?

- For a binomial distribution (as it is discrete) you could **rewrite all strict inequalities** (< and >) as **weak inequalities** (≤ and ≥) by using the identities for a binomial distribution
  - $P(X < x) = P(X \leq x - 1)$ and $P(X > x) = P(X \geq x + 1)$
  - For example: $P(X < 5) = P(X \leq 4)$ and $P(X > 5) = P(X \geq 6)$
- It helps to think about the **range of integers** you want
  - Identify the smallest and biggest integers in the range
- If your range has no minimum or maximum then use 0 or n
  - $P(X \leq b) = P(0 \leq X \leq b)$
  - $P(X \geq a) = P(a \leq X \leq n)$

- $P(a < X \leq b) = P(a + 1 \leq X \leq b)$
  - $P(5 < X \leq 9) = P(6 \leq X \leq 9)$

- $P(a \leq X < b) = P(a \leq X \leq b - 1)$
  - $P(5 \leq X < 9) = P(5 \leq X \leq 8)$

- $P(a < X < b) = P(a + 1 \leq X \leq b - 1)$
  - $P(5 < X < 9) = P(6 \leq X \leq 8)$

Worked example

The random variable $X \sim \mathrm{B}(40, 0.35)$. Find:

i) $\mathrm{P}(X = 10)$.

Identify n and p $n = 40$ $p = 0.35$

Use binomial probability distribution on GDC

$P(X = 10) = 0.057056...$

$P(X = 10) = 0.057$ (3sf)

ii) $\mathrm{P}(X \leq 10)$.

Identify upper and lower values

$P(X \leq 10) = P(0 \leq X \leq 10)$

Use binomial cumulative distribution on GDC

$P(X \leq 10) = 0.121491...$

$P(X \leq 10) = 0.121$ (3sf)

iii) $\mathrm{P}(8 < X < 15)$.

Identify upper and lower values

$P(8 < X < 15) = P(9 \leq X \leq 14)$

Use binomial cumulative distribution on GDC

$P(9 \leq X \leq 14) = 0.541827...$

$P(8 < X < 15) = 0.542$ (3sf)