



4.4 Probability Distributions

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4.4.1 Discrete Probability Distributions

Discrete Probability Distributions

What is a discrete random variable?

- A random variable is a variable whose value depends on the outcome of a random event
 - The value of the random variable is not known until the event is carried out (this is what is meant by 'random' in this case)
- Random variables are denoted using upper case letters (X, Y, etc.)
- Particular outcomes of the event are denoted using lower case letters (X, Y, etc)
- P(X=x) means "the probability of the random variable X taking the value X"
- A discrete random variable (often abbreviated to DRV) can only take certain values within a set
 - Discrete random variables usually count something
 - Discrete random variables usually can only take a finite number of values but it is possible that it can take an infinite number of values (see the examples below)
- **Examples** of discrete random variables include:
 - The number of times a coin lands on heads when flipped 20 times
 this has a finite number of outcomes: {0,1,2,...,20}
 - The number of emails a manager receives within an hour
 - this has an infinite number of outcomes: {1,2,3,...}
 - The number of times a dice is rolled until it lands on a 6
 - this has an infinite number of outcomes: {1,2,3,...}
 - The number that a dice lands on when rolled once
 - this has a finite number of outcomes: {1,2,3,4,5,6}

What is a probability distribution of a discrete random variable?

- A discrete probability distribution fully describes all the values that a discrete random variable can take along with their associated probabilities
 - This can be given in a **table**
 - Or it can be given as a **function** (called a discrete probability distribution function or "pdf")
 - They can be represented by **vertical line graphs** (the possible values for along the horizontal axis and the probability on the vertical axis)
- The sum of the probabilities of all the values of a discrete random variable is 1
 - This is usually written $\sum P(X=x) = 1$
- A **discrete uniform distribution** is one where the random variable takes a finite number of values each with an **equal probability**
 - If there are n values then the probability of each one is



How do I calculate probabilities using a discrete probability distribution?

- First draw a table to represent the probability distribution
 - If it is given as a function then find each probability
 - If any probabilities are unknown then use algebra to represent them
- Form an equation using $\sum P(X=x) = 1$
 - Add together all the probabilities and make the sum equal to 1
- To find P(X=k)
 - If k is a possible value of the random variable X then P(X = k) will be given in the table
 - If k is not a possible value then P(X=k)=0
- To find $P(X \le k)$
 - Identify all possible values, X_i , that X can take which satisfy $X_i \le k$
 - Add together all their corresponding probabilities
 - $P(X \le k) = \sum_{x_i \le k} P(X = x_i)$
 - Some mathematicians use the notation F(x) to represent the cumulative distribution

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$$F(x) = P(X \le x)$$

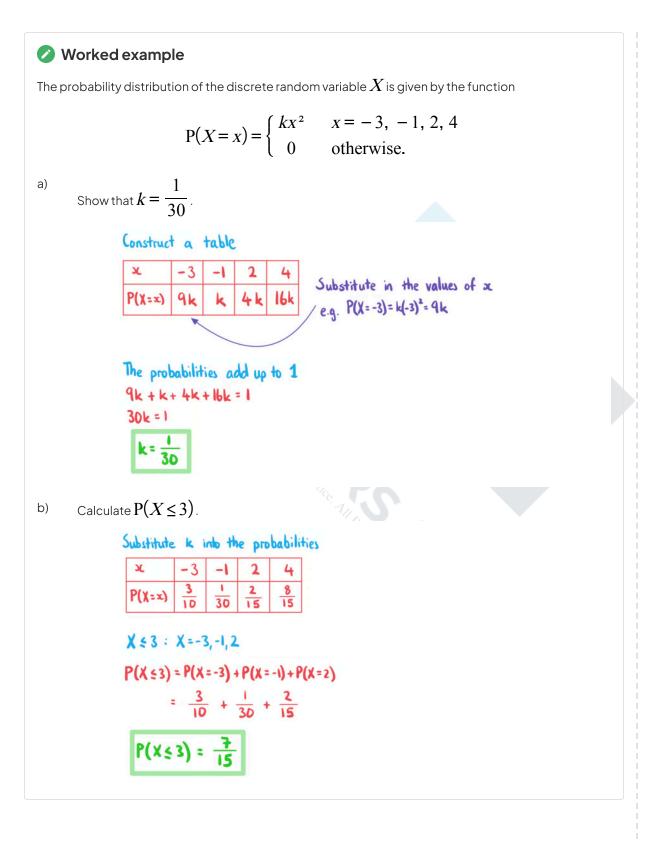
- Using a similar method you can find P(X < k), P(X > k) and $P(X \ge k)$
- As all the probabilities add up to 1 you can form the following equivalent equations:
 - P(X < k) + P(X = k) + P(X > k) = 1
 - $P(X > k) = 1 P(X \le k)$
 - $P(X \ge k) = 1 P(X < k)$

How do I know which inequality to use?

- P(X≤k) would be used for phrases such as:
 At most, no greater than, etc
- P(X < k) would be used for phrases such as:
 Fewer than
- $P(X \ge k)$ would be used for phrases such as:
 - At least , no fewer than , etc
- P(X > k) would be used for phrases such as:
 - Greater than , etc

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4.4.2 Mean & Variance

Expected Values E(X)

What does E(X) mean and how do I calculate E(X)?

- E(X) means the expected value or the mean of a random variable X
 - The expected value does not need to be an obtainable value of X
 - For example: the expected value number of times a coin will land on tails when flipped 5 times is 2.5
- For a **discrete** random variable, it is calculated by:
 - Multiplying each value of X with its corresponding probability
 - Adding all these terms together

$$E(X) = \sum x P(X = x)$$

- This is given in the **formula booklet**
- Look out for symmetrical distributions (where the values of X are symmetrical and their probabilities are symmetrical) as the mean of these is the same as the median
 - For example: if X can take the values 1, 5, 9 with probabilities 0.3, 0.4, 0.3 respectively then by symmetry the mean would be 5

How can I decide if a game is fair?

- Let X be the random variable that represents the gain/loss of a player in a game
 - X will be **negative** if there is a **loss**
- Normally the expected gain or loss is calculated by subtracting the cost to play the game from the expected value of the prize
- If E(X) is **positive** then it means the player can **expect to make a gain**
- If E(X) is negative then it means the player can expect to make a loss Parch Cfi
- The game is called fair if the expected gain is 0
 - E(X) = 0



Worked example

Daphne pays \$15 to play a game where she wins a prize of \$1, \$5, \$10 or \$100. The random variable W represents the amount she wins and has the probability distribution shown in the following table:

W	1	5	10	100
P(W=w)	0.35	0.5	0.05	0.1

a) Calculate the expected value of Daphne's prize.

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 $E(W) = \sum w P(W = w)$ = 1 × 0.35 + 5 × 05 + 10 × 0.05 + 100 × 0.1

b) Determine whether the game is fair.

A game is fair is expected gain/loss is 0 Prize - Cost 13.35 -15 = -1.65

Expected loss is \$1.65 so game is not fair

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Variance Var(X)

What does Var(X) mean and how do I calculate Var(X)?

- Var(X) means the variance of a random variable X
 - The standard deviation is the square root of the variance
 - This provides a measure of the spread of the outcomes of X
 - The variance and standard deviation can **never be negative**
- The variance of X is the mean of the squared difference between X and the mean

$$Var(X) = E(X - \mu)^2$$

- This is given in the formula booklet
- This formula can be rearranged into the more useful form:

$Var(X) = E(X^2) - [E(X)]^2$

- This is given in the formula booklet
 - Compare this formula to the formula for the variance of a set of data
- This formula works for both discrete and continuous X

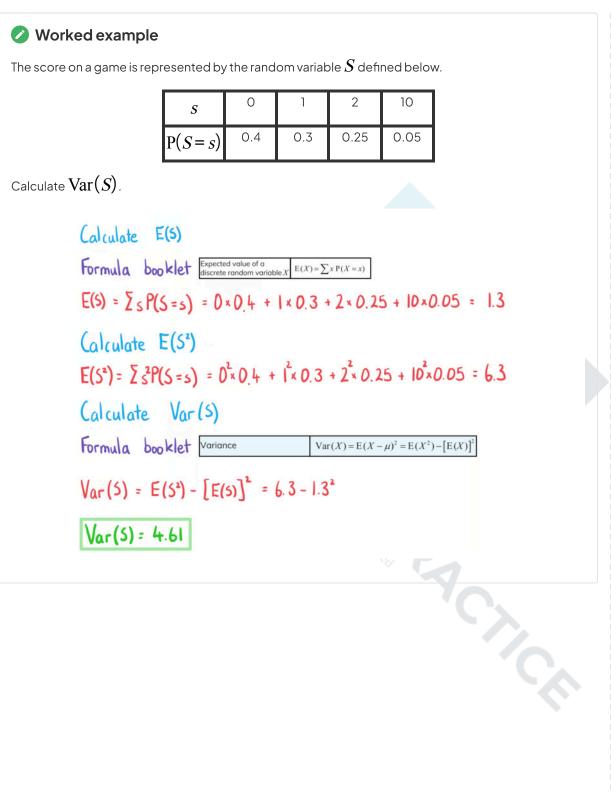
How do I calculate E(X²) for discrete X?

- E(X²) means the expected value or the mean of the random variable defined as X²
- For a discrete random variable, it is calculated by:
 - Squaring each value of X to get the values of X²
 - Multiplying each value of X² with its corresponding probability
 - Adding all these terms together
 - $E(X^2) = \sum X^2 P(X = X)$
 - This is given in the formula booklet as part of the formula for Var(X)
 - $Var(X) = \sum X^2 P(X = x) \mu^2$
- E(f(X)) can be found in a similar way

Is E(X²) equal to E(X)²?

- Definitely not!
 - They are only equal if X can only take one value
- E(X²) is the mean of the values of X²
- E(X)² is the square of the mean of the values of X
- To see the difference
- Parchice. Imagine a random variable X that can only take 1 and -1 with equal chance
 - E(X) = 0 so E(X)² = 0
 - The square values are 1 and 1 so E(X²) = 1







Transformation of a Single Variable

How do I calculate the expected value and variance of a transformation of X?

- Suppose X is transformed by the function f to form a new variable T = f(X)
 This means the function f is applied to all possible values of X
- Create a new probability distribution table
 - The top row contains the values $t_i = f(x_i)$
 - The bottom row still contains the values $P(X = x_i)$ which are unchanged as:

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$$P(X = x_i) = P(f(X) = f(x_i)) = P(T = t_i)$$

- Some values of T may be equal so you can add their probabilities together
- The **mean** is calculated in the same way
 - $E(T) = \sum t P(X = x)$
- The variance is calculated using the same formula
 - $Var(T) = E(T^2) [E(T)]^2$

Are there any shortcuts?

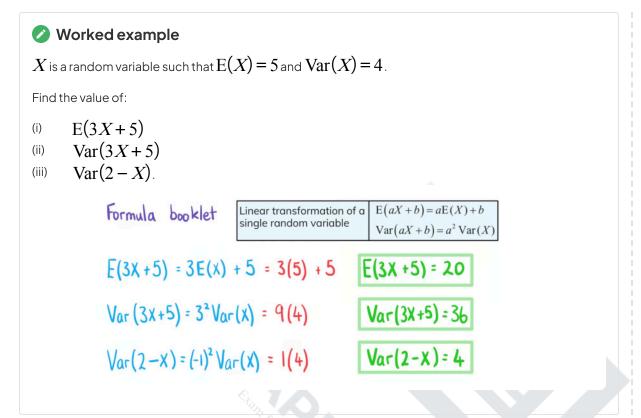
- There are formulae which can be used if the transformation is linear
 - T = aX + b where a and b are constants
- If the transformation is not linear then there are no shortcuts
 - You will have to first find the probability distribution of T

What are the formulae for E(aX + b) and Var(aX + b)?

- If a and b are constants then the following formulae are true:
 - $\bullet E(aX + b) = aE(X) + b$
 - $\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$
 - These are given in the formula booklet
- This is the same as linear transformations of data
 - The mean is affected by multiplication and addition/subtraction
 - The variance is affected by multiplication but not addition/subtraction
- Remember division can be written as a multiplication

$$\frac{X}{a} = \frac{1}{a}X$$





Practice

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