EXAM PAPERS PRACTICE

Boost your performance and confidence with these topic-based exam questions

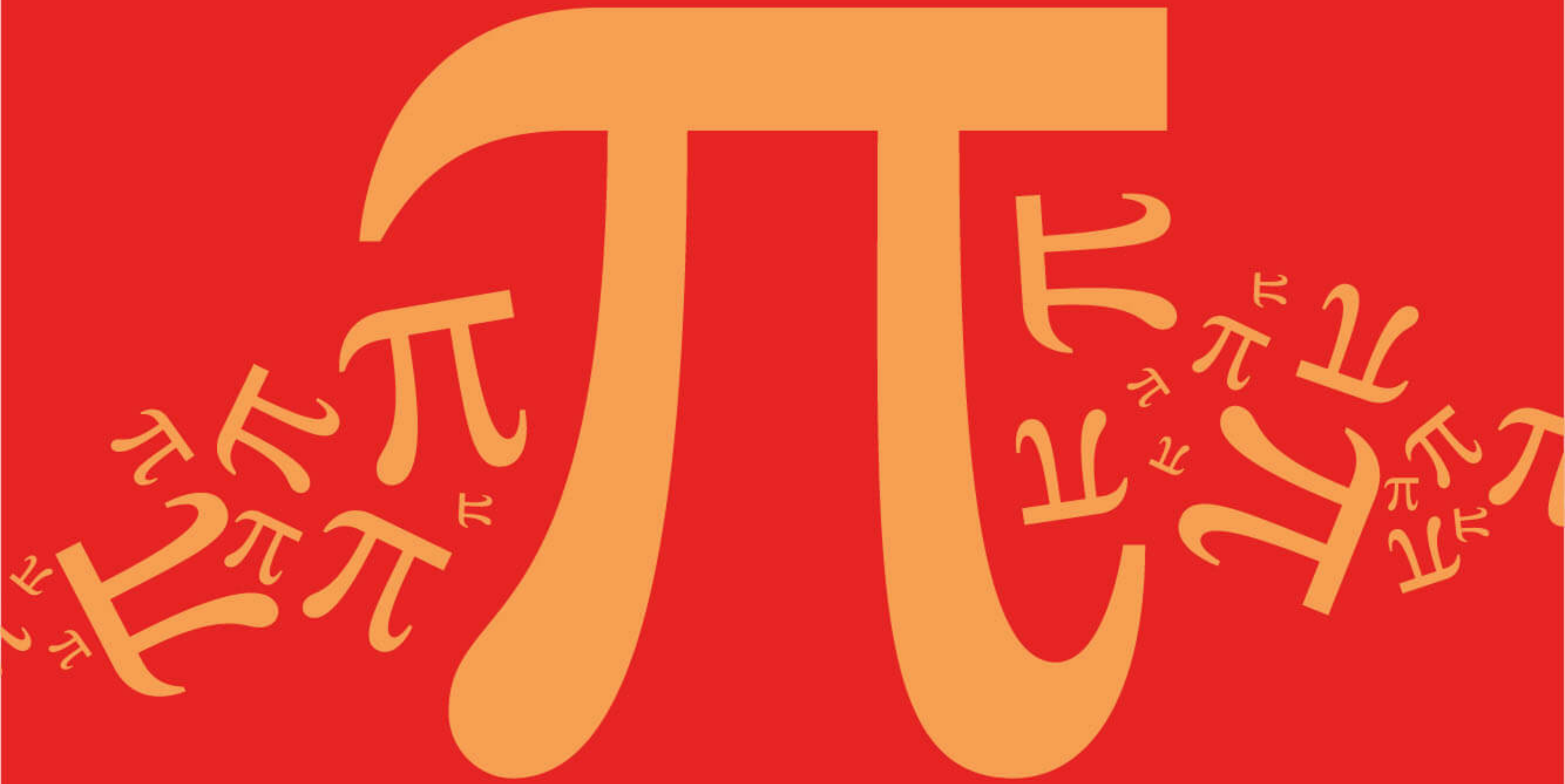
Practice questions created by actual examiners and assessment experts

Detailed mark scheme

Suitable for all boards

Designed to test your ability and thoroughly prepare you

4.4 Probability



IB Maths - Revision Notes

4.4.1 Probability & Types of Events

Probability Basics

What key words and terminology are used with probability?

- An **experiment** is a repeatable activity that has a result that can be observed or recorded
  - **Trials** are what we call the repeats of the experiment
- An **outcome** is a possible result of a trial
- An **event** is an outcome or a collection of outcomes
  - Events are usually denoted with capital letters: A, B, etc
  - $n(A)$ is the number of outcomes that are included in event A
  - An event can have one or more than one outcome
- A **sample space** is the set of all possible outcomes of an experiment
  - This is denoted by U
  - $n(U)$ is the total number of outcomes
  - It can be represented as a **list** or a **table**

How do I calculate basic probabilities?

- If all outcomes are **equally likely** then probability for each outcome is the same
  - Probability for each outcome is $\frac{1}{n(U)}$
- **Theoretical probability** of an event can be calculated without using an experiment by dividing the number of outcomes of that event by the total number of outcomes

$$P(A) = \frac{n(A)}{n(U)}$$

  - This is given in the **formula booklet**
  - Identifying all possible outcomes either as a list or a table can help
- **Experimental probability** (also known as **relative frequency**) of an outcome can be calculated using results from an experiment by dividing its frequency by the number of trials
  - **Relative frequency** of an outcome is $\frac{\text{Frequency of that outcome from the trials}}{\text{Total number of trials } (n)}$

How do I calculate the expected number of occurrences of an outcome?

- **Theoretical probability** can be used to calculate the **expected number of occurrences** of an outcome from n trials
- If the probability of an outcome is p and there are n trials then:
  - The expected number of occurrences is np

- This **does not mean** that there will **exactly *np* occurrences**
- If the experiment is repeated multiple times then we expect the number of occurrences to average out to be *np*

What is the complement of an event?

- The probabilities of all the outcomes **add up to 1**
- Complementary events are when there are **two events** and **exactly one** of them will occur
  - One event has to occur but both events can not occur at the same time
- The **complement of event *A*** is the event where event ***A* does not happen**
  - This can be thought of as **not *A***
  - This is denoted A'

$$\mathrm{P}(A) + \mathrm{P}(A') = 1$$

  - This is in the **formula booklet**
  - It is commonly written as $\mathrm{P}(A') = 1 - \mathrm{P}(A)$

What are different types of combined events?

- The **intersection** of two events (*A* and *B*) is the event where **both *A* and *B*** occur
  - This can be thought of as ***A* and *B***
  - This is denoted as $A \cap B$
- The **union** of two events (*A* and *B*) is the event where ***A* or *B* or both occur**
  - This can be thought of as ***A* or *B***
  - This is denoted $A \cup B$
- The event where *A* occurs given that event *B* has occurred is called **conditional probability**
  - This can be thought of as ***A* given *B***
  - This is denoted $A | B$

How do I find the probability of combined events?

- The probability of *A* **or** *B* (or both) occurring can be found using the formula

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

  - This is given in the **formula booklet**
  - You subtract the probability of *A* and *B* both occurring because it has been included twice (once in P(*A*) and once in P(*B*))
- The probability of *A* **and** *B* occurring can be found using the formula

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A)$$

  - A rearranged version is given in the **formula booklet**
  - Basically you multiply the probability of *A* by the probability of *B* then happening

Exam Tip

- In an exam drawing a Venn diagram or tree diagram can help even if the question does not ask you to

Worked example

Dave has two fair spinners, A and B. Spinner A has three sides numbered 1, 4, 9 and spinner B has four sides numbered 2, 3, 5, 7. Dave spins both spinners and forms a two-digit number by using the spinner A for the first digit and spinner B for the second digit.

T is the event that the two-digit number is a multiple of 3.

a) List all the possible two-digit numbers.

A two-way table would be a systematic way to list all the outcomes

	2	3	5	7
1	12	13	15	17
4	42	43	45	47
9	92	93	95	97

b) Find $\text{P}(T)$.

$P(T) = \frac{n(T)}{n(U)}$ ← Number of multiples of 3 / ← Total number of outcomes

$\{12, 15, 42, 45, 93\}$ are the multiples of 3

$P(T) = \frac{5}{12}$

c) Find $\text{P}(T')$.

$P(T) + P(T') = 1 \Rightarrow P(T') = 1 - P(T)$

$P(T') = 1 - \frac{5}{12}$

$P(T') = \frac{7}{12}$

Independent & Mutually Exclusive Events

What are mutually exclusive events?

- Two events are **mutually exclusive** if they **cannot both occur**
  - For example: when rolling a dice the events "getting a prime number" and "getting a 6" are mutually exclusive
- If *A* and *B* are mutually exclusive events then:
  - $\mathrm{P}(A \cap B) = 0$

What are independent events?

- Two events are **independent** if **one occurring does not affect the probability of the other occurring**
  - For example: when flipping a coin twice the events "getting a tails on the first flip" and "getting a tails on the second flip" are independent
- If *A* and *B* are independent events then:
  - $\mathrm{P}(A|B) = \mathrm{P}(A)$ and $\mathrm{P}(B|A) = \mathrm{P}(B)$
- If *A* and *B* are independent events then:
  - $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
    - This is given in the **formula booklet**
    - This is a useful formula to test whether two events are statistically independent

How do I find the probability of combined mutually exclusive events?

- If *A* and *B* are **mutually exclusive** events then

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

  - This is given in the **formula booklet**
  - This occurs because $\mathrm{P}(A \cap B) = 0$
- For any two events *A* and *B* the events $A \cap B$ and $A \cap B'$ are **mutually exclusive** and *A* is the **union** of these two events
  - $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B')$
    - This works for any two events *A* and *B*

Worked example

a) A student is chosen at random from a class. The probability that they have a dog is 0.8, the probability they have a cat is 0.6 and the probability that they have a cat or a dog is 0.9. Find the probability that the student has both a dog and a cat.

Let D be event "has a dog" and C be "has a cat"

$P(D \cup C) = P(D) + P(C) - P(D \cap C)$

$0.9 = 0.8 + 0.6 - P(D \cap C)$

$P(D \cap C) = 0.5$

b) Two events, Q and R, are such that $\mathrm{P}(Q) = 0.8$ and $\mathrm{P}(Q \cap R) = 0.1$. Given that Q and R are independent, find $\mathrm{P}(R)$.

Q and R independent $\Rightarrow P(Q \cap R) = P(Q)P(R)$

$0.1 = 0.8 \times P(R) \quad \therefore \quad P(R) = \frac{0.1}{0.8}$

$P(R) = 0.125$ or $\frac{1}{8}$

c) Two events, S and T, are such that $\mathrm{P}(S) = 2\mathrm{P}(T)$. Given that S and T are mutually exclusive and that $\mathrm{P}(S \cup T) = 0.6$ find $\mathrm{P}(S)$ and $\mathrm{P}(T)$.

S and T mutually exclusive $\Rightarrow P(S \cup T) = P(S) + P(T)$

$0.6 = P(S) + P(T)$

$0.6 = 2P(T) + P(T)$ $\quad P(S) = 2P(T)$

$0.6 = 3P(T)$

$P(T) = 0.2$ and $P(S) = 0.4$

4.4.2 Conditional Probability

Conditional Probability

What is conditional probability?

- **Conditional probability** is where the probability of an **event** happening can vary depending on the outcome of a prior event
- The event A happening **given that** event B has happened is denoted $A|B$
- A common example of conditional probability involves selecting multiple objects from a bag **without replacement**
  - The probability of selecting a certain item changes depending on what was selected before
    - This is because the total number of items will change as they are not replaced once they have been selected

How do I calculate conditional probabilities?

- Some conditional probabilities can be calculated by using counting outcomes
  - Probabilities without replacement can be calculated like this
  - For example: There are 10 balls in a bag, 6 of them are red, two of them are selected without replacement
    - To find the probability that the second ball selected is red given that the first one is red count how many balls are left:
    - A red one has already been selected so there are 9 balls left and 5 are red so the probability is $\frac{5}{9}$
- You can use sample space diagrams to find the probability of A given B:
  - reduce your sample space to just include outcomes for event B
  - find the proportion that also contains outcomes for event A
- There is a formula for conditional probability that you can use
  - $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
  - This is given in the **formula booklet**
  - This can be rearranged to give $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A|B)$

Worked example

In a class of 30 students: 19 students have a dog, 17 students have a cat and 11 have both a dog and a cat. One student is selected at random.

a) Find the probability that the student has a dog.

Let D be event "has a dog" and C be "has a cat"

$P(D) = \frac{n(D)}{n(U)}$ ← Number who have dogs / ← Total number of students

$P(D) = \frac{19}{30}$

b) Find the probability that the student has a dog given that they have a cat.

17 have a cat of which 11 also have a dog

$P(D|C) = \frac{11}{17}$

Could also use $P(D|C) = \frac{P(D \cap C)}{P(C)}$

c) Find the probability that the student has a cat given that they have a dog.

19 have a dog of which 11 also have a cat

$P(C|D) = \frac{11}{19}$

Could also use $P(C|D) = \frac{P(C \cap D)}{P(D)}$

4.4.3 Sample Space Diagrams

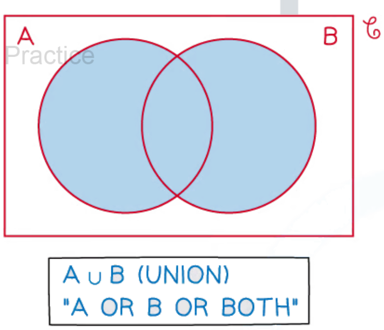
Venn Diagrams

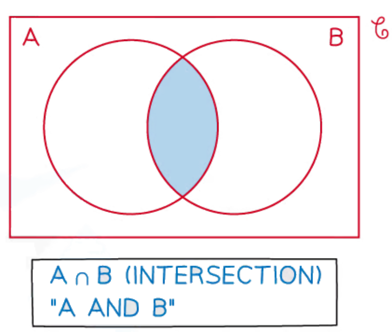
What is a Venn diagram?

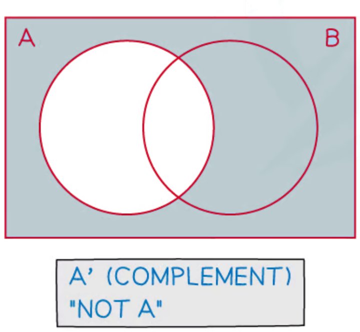
- A Venn diagram is a way to illustrate **events** from an **experiment** and are particularly useful when there is an overlap between possible **outcomes**
- A Venn diagram consists of
  - a **rectangle** representing the **sample space (*U*)**
    - The rectangle is labelled U
    - Some mathematicians instead use S or ξ
  - a **circle** for each **event**
    - Circles may or may not overlap depending on which **outcomes** are shared between **events**
- The numbers in the circles represent either the **frequency** of that event or the **probability** of that event
  - If the **frequencies** are used then they should **add up to the total frequency**
  - If the **probabilities** are used then they should **add up to 1**

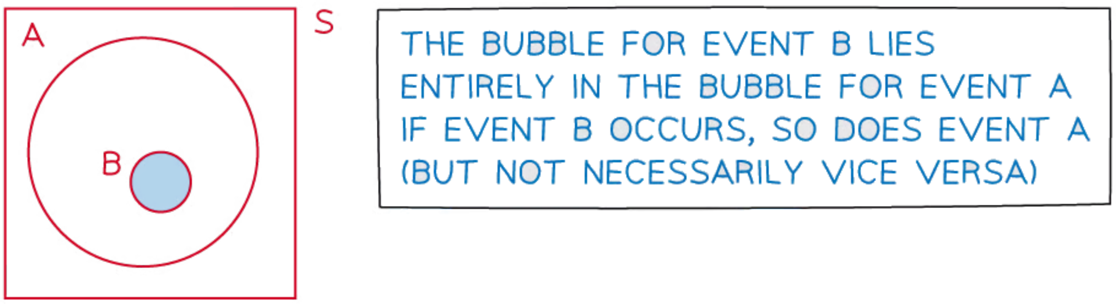
What do the different regions mean on a Venn diagram?

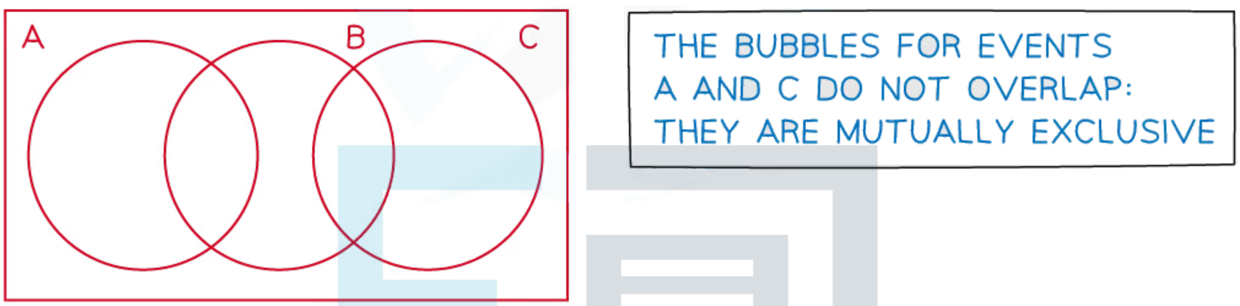
- A' is represented by the regions that are **not in** the A circle
- $A \cap B$ is represented by the region where the A and B circles **overlap**
- $A \cup B$ is represented by the regions that **are in** A or B or both
- Venn diagrams show '**AND**' and '**OR**' statements easily
- Venn diagrams also instantly show **mutually exclusive** events as these circles will **not overlap**
- **Independent** events cannot be instantly seen
  - You need to use probabilities to deduce if two events are independent





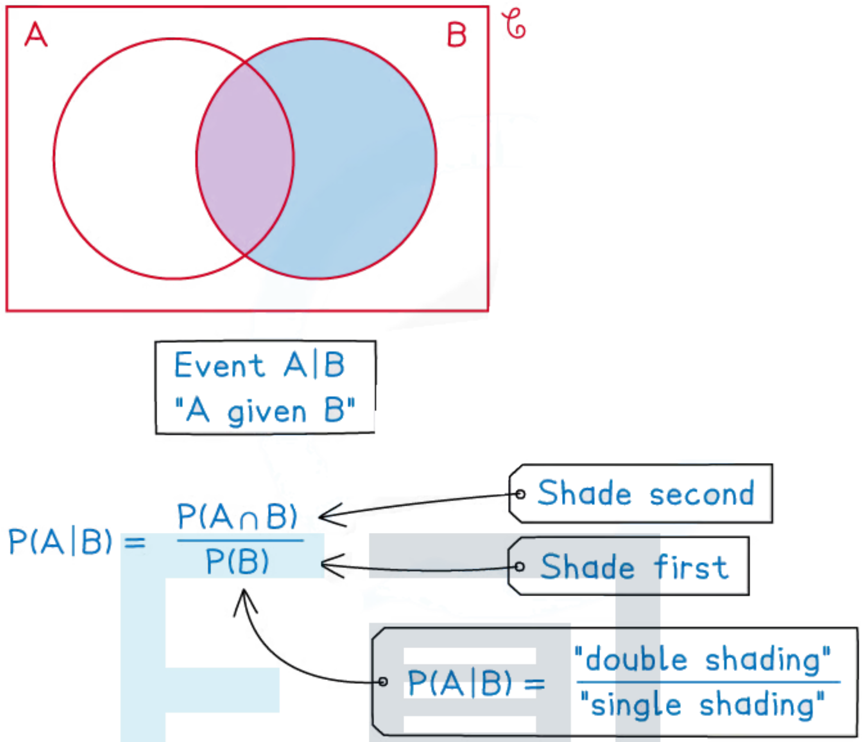






How do I solve probability problems involving Venn diagrams?

- Draw, or add to a given Venn diagram, filling in as many values as possible from the information provided in the question
- It is usually helpful to work from the centre outwards
  - Fill in **intersections** (overlaps) first
- If two events are independent you can use the formula
  - $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
- To find the conditional probability $\mathrm{P}(A \mid B)$
  - Add together the frequencies/probabilities in the *B* circle
    - This is your denominator
  - Out of those frequencies/probabilities add together the ones that are also in the *A* circle
    - This is your numerator
  - Evaluate the fraction



Exam Tip

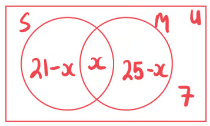
- If you struggle to fill in a Venn diagram in an exam:
  - Label the missing parts using algebra
  - Form equations using known facts such as:
    - the sum of the probabilities should be 1
    - $P(A \cap B) = P(A)P(B)$ if A and B are independent events

Worked example

40 people are asked if they have sugar and/or milk in their coffee. 21 people have sugar, 25 people have milk and 7 people have neither.

a) Draw a Venn diagram to represent the information.

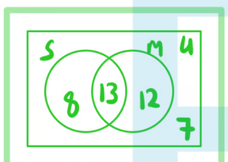
Find the centre first



Total should be 40

$(21-x)+x+(25-x)+7=40$

$53-x=40 \quad \therefore \ x=13$



b) One of the 40 people are randomly selected, find the probability that they have sugar but not milk with their coffee.

S and not M is the part of S circle that does not include M

$P(S \cap M') = \frac{8}{40}$

Remember to write as a fraction of the total

$P(S \cap M') = \frac{1}{5}$

c) Given that a person who has sugar is selected at random, find the probability that they have milk with their coffee.

Given that sugar has been selected we only want the S circle as our total.

Out of the S circle 13 also have milk

$P(M|S) = \frac{13}{21}$

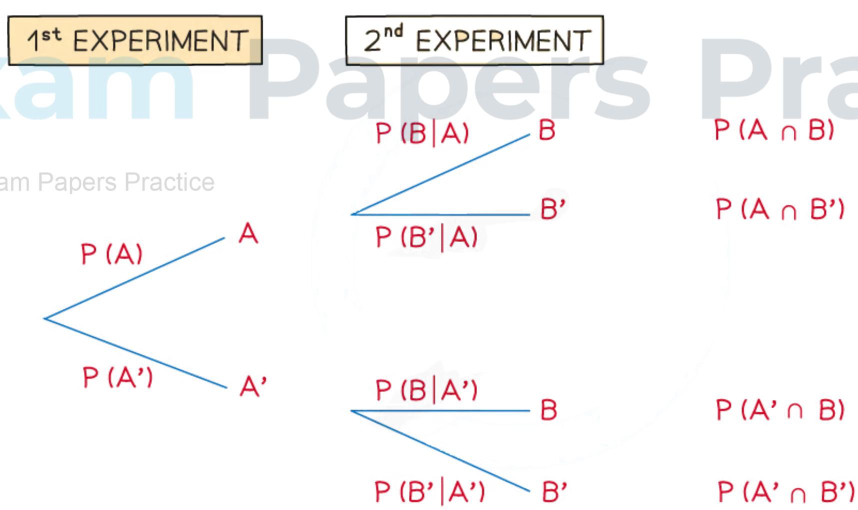
Tree Diagrams

What is a tree diagram?

- A **tree diagram** is another way to show the outcomes of combined events
  - They are very useful for intersections of events
- The events on the branches must be **mutually exclusive**
  - Usually they are an event and its complement
- The probabilities on the second sets of branches **can depend** on the outcome of the first event
  - These are **conditional probabilities**
- When selecting the items from a bag:
  - The second set of branches will be the **same** as the first if the items **are replaced**
  - The second set of branches will be the **different** to the first if the items **are not replaced**

How are probabilities calculated using a tree diagram?

- To find the probability that two events happen together you **multiply** the corresponding probabilities on their branches
  - It is helpful to find the probability of all combined outcomes once you have drawn the tree
- To find the probability of an event you can:
  - **add together** the probabilities of the **combined outcomes** that are part of that event
    - For example: $\mathrm{P}(A \cup B) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B)$
  - **subtract** the probabilities of the combined outcomes that are not part of that event from 1
    - For example: $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(A' \cap B')$



Do I have to use a tree diagram?

- If there are **multiple events** or trials then a tree diagram can get big
- You can break down the problem by using the words **AND/OR/NOT** to help you find probabilities without a tree
- You can speed up the process by only drawing parts of the tree that you are interested in

Which events do I put on the first branch?

- If the events A and B are **independent** then the **order does not matter**
- If the events A and B are **not independent** then the **order does matter**
  - If you have the probability of ***A* given *B*** then put ***B* on the first set** of branches
  - If you have the probability of ***B* given *A*** then put ***A* on the first set** of branches

Exam Tip

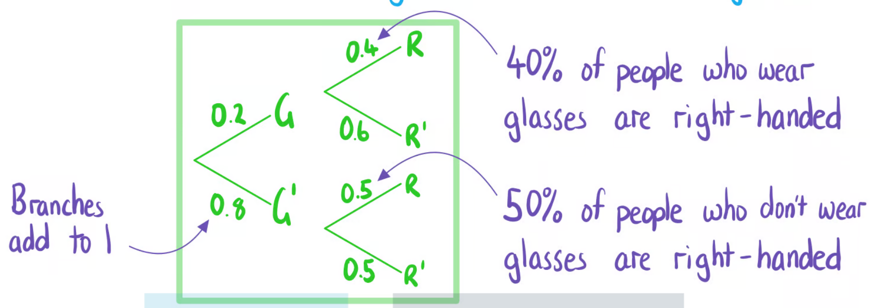
- In an exam do not waste time drawing a full tree diagram for scenarios with lots of events unless the question asks you to
  - Only draw the parts that you are interested in

Worked example

20% of people in a company wear glasses. 40% of people in the company who wear glasses are right-handed. 50% of people in the company who don't wear glasses are right-handed.

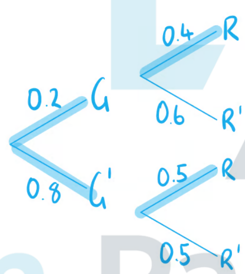
a) Draw a tree diagram to represent the information.

Let G be the event "wears glasses" and R be "is right-handed"



b) One of the people in the company are randomly selected, find the probability that they are right-handed.

Find options that contain R



$P(G \cap R) = 0.2 \times 0.4 = 0.08$

Multiply along branches

$P(G' \cap R) = 0.8 \times 0.5 = 0.4$

$P(R) = P(G \cap R) + P(G' \cap R') = 0.08 + 0.4$

$P(R) = 0.48$

c) Given that a person who is right-handed is selected at random, find the probability that they wear glasses.

$$P(G|R) = \frac{P(G \cap R)}{P(R)} = \frac{0.08}{0.48}$$

$$P(G|R) = \frac{1}{6}$$