4.4 Probability Distributions

IB Maths - Revision Notes

AA SL

Exam Papers Practice

4.4.1 Discrete Probability Distributions

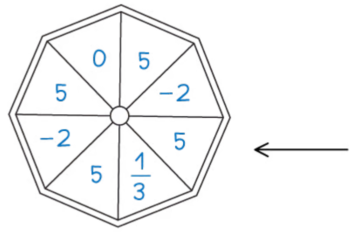
Discrete Probability Distributions

What is a discrete random variable?

- A **random variable** is a variable whose value depends on the outcome of a **random event**
  - The value of the random variable is not known until the event is carried out (this is what is meant by 'random' in this case)
- **Random variables** are denoted using **upper case letters** (X, Y, etc)
- **Particular outcomes** of the event are denoted using **lower case letters** (x, y, etc)
- $\mathrm{P}(X=x)$ means "the probability of the random variable X taking the value x"
- A **discrete** random variable (often abbreviated to DRV) can only take **certain values** within a set
  - Discrete random variables **usually count** something
  - Discrete random variables usually can only take a finite number of values but it is possible that it can take an infinite number of values (see the examples below)
- **Examples** of discrete random variables include:
  - The number of times a coin lands on heads when flipped 20 times
    - this has a finite number of outcomes: {0,1,2,…,20}
  - The number of emails a manager receives within an hour
    - this has an infinite number of outcomes: {1,2,3,…}
  - The number of times a dice is rolled until it lands on a 6
    - this has an infinite number of outcomes: {1,2,3,…}
  - The number that a dice lands on when rolled once
    - this has a finite number of outcomes: {1,2,3,4,5,6}

What is a probability distribution of a discrete random variable?

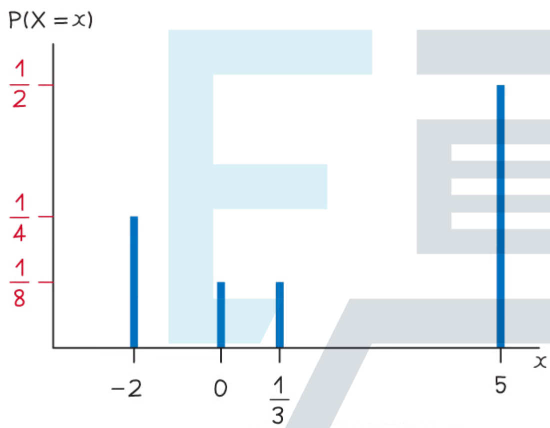
- A **discrete probability distribution** fully describes **all the values** that a discrete random variable can take along with their **associated probabilities**
  - This can be given in a **table**
  - Or it can be given as a **function** (called a discrete probability distribution function or "pdf")
  - They can be represented by **vertical line graphs** (the possible values for along the horizontal axis and the probability on the vertical axis)
- The **sum of the probabilities** of **all the values** of a discrete random variable is **1**
  - This is usually written $\sum \mathrm{P}(X=x)=1$
- A **discrete uniform distribution** is one where the random variable takes a finite number of values each with an **equal probability**
  - If there are n values then the probability of each one is $\frac{1}{n}$



LET x BE THE NUMBER THAT THE SPINNER LANDS ON

x	-2	0	$\frac{1}{3}$	5
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$

$$P(X=x)=\begin{cases}\frac{1}{8} & x=0,\ \frac{1}{3}\\ \frac{1}{4} & x=-2\\ \frac{1}{2} & x=5\\ 0 & \text{OTHERWISE}\end{cases}$$



How do I calculate probabilities using a discrete probability distribution?

- First **draw a table** to represent the probability distribution
  - If it is given as a function then find each probability
  - If any probabilities are unknown then use algebra to represent them
- **Form an equation** using $\sum P(X=x)=1$
  - Add together all the probabilities and make the sum equal to 1
- To find $P(X=k)$
  - If k is a possible value of the random variable X then $P(X=k)$ will be given in the table
  - If k is not a possible value then $P(X=k)=0$
- To find $P(X\le k)$
  - Identify all possible values, x_i, that X can take which satisfy $x_i \le k$
  - Add together all their corresponding probabilities
  - $P(X\le k)=\sum_{x_i\le k} P(X=x_i)$
  - Some mathematicians use the notation $F(x)$ to represent the cumulative distribution
    - $F(x)=P(X\le x)$
- Using a similar method you can find $P(X<k)$, $P(X>k)$ and $P(X\ge k)$

- As all the probabilities add up to 1 you can form the following equivalent equations:
  - $P(X < k) + P(X = k) + P(X > k) = 1$
  - $P(X > k) = 1 - P(X \leq k)$
  - $P(X \geq k) = 1 - P(X < k)$

How do I know which inequality to use?

- $P(X \leq k)$ would be used for phrases such as:
  - At most, no greater than, etc
- $P(X < k)$ would be used for phrases such as:
  - Fewer than
- $P(X \geq k)$ would be used for phrases such as:
  - At least, no fewer than, etc
- $P(X > k)$ would be used for phrases such as:
  - Greater than, etc

Worked example

The probability distribution of the discrete random variable X is given by the function

$$P(X=x)=\begin{cases} kx^2 & x=-3,\ -1,\ 2,\ 4 \\ 0 & \text{otherwise.} \end{cases}$$

a) Show that $k=\frac{1}{30}$.

Construct a table

x	-3	-1	2	4
$P(X=x)$	$9k$	k	$4k$	$16k$

Substitute in the values of x

e.g. $P(X=-3)=k(-3)^2=9k$

The probabilities add up to 1

$9k+k+4k+16k=1$

$30k=1$

$k=\frac{1}{30}$

b) Calculate $P(X\le 3)$.

Substitute k into the probabilities

x	-3	-1	2	4
$P(X=x)$	$\frac{3}{10}$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{8}{15}$

$X\le 3$: $X=-3,-1,2$

$P(X\le 3)=P(X=-3)+P(X=-1)+P(X=2)$

$=\frac{3}{10}+\frac{1}{30}+\frac{2}{15}$

$P(X\le 3)=\frac{7}{15}$

4.4.2 Expected Values

Expected Values E(X)

What does E(X) mean and how do I calculate E(X)?

- **E(X)** means the **expected value** or the **mean** of a **random variable X**
  - The expected value does not need to be an obtainable value of X
  - For example: the expected value number of times a coin will land on tails when flipped 5 times is 2.5
- For a **discrete** random variable, it is calculated by:
  - **Multiplying each value** of X with its corresponding **probability**
  - **Adding** all these terms together

$$E(X) = \sum x P(X = x)$$

    - This is given in the **formula booklet**
- Look out for **symmetrical** distributions (where the values of X are symmetrical and their probabilities are symmetrical) as the mean of these is the same as the median
  - For example: if X can take the values 1, 5, 9 with probabilities 0.3, 0.4, 0.3 respectively then by symmetry the mean would be 5

How can I decide if a game is fair?

- Let X be the random variable that represents the **gain/loss** of a player in a game
  - X will be **negative** if there is a **loss**
- Normally the expected gain or loss is calculated by **subtracting** the **cost to play** the game from the **expected value** of the **prize**
- If $E(X)$ is **positive** then it means the player can **expect to make a gain**
- If $E(X)$ is **negative** then it means the player can **expect to make a loss**
- The game is called **fair** if the **expected gain is 0**
  - $E(X) = 0$

Worked example

Daphne pays $15 to play a game where she wins a prize of $1, $5, $10 or $100. The random variable W represents the amount she wins and has the probability distribution shown in the following table:

w	1	5	10	100
$P(W = w)$	0.35	0.5	0.05	0.1

a) Calculate the expected value of Daphne's prize.

Formula booklet

Expected value of a discrete random variable X	$E(X) = \sum x P(X = x)$

$E(W) = \sum w P(W = w)$

$= 1 \times 0.35 + 5 \times 0.5 + 10 \times 0.05 + 100 \times 0.1$

Expected value = $13.35

b) Determine whether the game is fair.

A game is fair is expected gain/loss is 0

Prize − cost

$13.35 - 15 = -1.65$

Expected loss is $1.65 so game is not fair