DP IB Maths: AA HL

4.3 Probability

Contents

4.3.1 Probability & Types of Events

Probability Basics

What key words and terminology are used with probability?

- An **experiment** is a repeatable activity that has a result that can be observed or recorded
  - **Trials** are what we call the repeats of the experiment
- An **outcome** is a possible result of a trial
- An **event** is an outcome or a collection of outcomes
  - Events are usually denoted with capital letters: *A*, *B*, etc
  - $n(A)$ is the number of outcomes that are included in event *A*
  - An event can have one or more than one outcome
- A **sample space** is the set of all possible outcomes of an experiment
  - This is denoted by *U*
  - $n(U)$ is the total number of outcomes
  - It can be represented as a **list** or a **table**

How do I calculate basic probabilities?

- If all outcomes are **equally likely** then probability for each outcome is the same
  - Probability for each outcome is $\frac{1}{n(U)}$
- **Theoretical probability** of an event can be calculated without using an experiment by dividing the number of outcomes of that event by the total number of outcomes

$$P(A) = \frac{n(A)}{n(U)}$$

  - This is given in the **formula booklet**
  - Identifying all possible outcomes either as a list or a table can help
- **Experimental probability** (also known as **relative frequency**) of an outcome can be calculated using results from an experiment by dividing its frequency by the number of trials
  - **Relative frequency** of an outcome is $\frac{\text{Frequency of that outcome from the trials}}{\text{Total number of trials } (n)}$

How do I calculate the expected number of occurrences of an outcome?

- **Theoretical probability** can be used to calculate the **expected number of occurrences** of an outcome from *n* trials
- If the probability of an outcome is *p* and there are *n* trials then:
  - The expected number of occurrences is ***np***
  - This **does not mean** that there will **exactly *np* occurrences**

- If the experiment is repeated multiple times then we expect the number of occurrences to average out to be np

What is the complement of an event?

- The probabilities of all the outcomes **add up to 1**
- Complementary events are when there are **two events** and **exactly one** of them will occur
  - One event has to occur but both events can not occur at the same time
- The **complement of event *A*** is the event where event ***A* does not happen**
  - This can be thought of as **not *A***
  - This is denoted A'

$$\mathrm{P}(A) + \mathrm{P}(A') = 1$$

    - This is in the **formula booklet**
    - It is commonly written as $\mathrm{P}(A') = 1 - \mathrm{P}(A)$

What are different types of combined events?

- The **intersection** of two events (A and B) is the event where **both *A* and *B*** occur
  - This can be thought of as ***A* and *B***
  - This is denoted as $A \cap B$
- The **union** of two events (A and B) is the event where ***A* or *B* or both occur**
  - This can be thought of as ***A* or *B***
  - This is denoted $A \cup B$
- The event where A occurs given that event B has occurred is called **conditional probability**
  - This can be thought as ***A* given *B***
  - This is denoted $A|B$

How do I find the probability of combined events?

- The probability of A **or** B (or both) occurring can be found using the formula

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

    - This is given in the **formula booklet**
    - You subtract the probability of A and B both occurring because it has been included twice (once in P(A) and once in P(B))
- The probability of A **and** B occurring can be found using the formula

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A)$$

    - A rearranged version is given in the **formula booklet**
    - Basically you multiply the probability of A by the probability of B then happening

Worked example

Dave has two fair spinners, *A* and *B*. Spinner *A* has three sides numbered 1, 4, 9 and spinner *B* has four sides numbered 2, 3, 5, 7. Dave spins both spinners and forms a two-digit number by using the spinner *A* for the first digit and spinner *B* for the second digit.

T is the event that the two-digit number is a multiple of 3.

a) List all the possible two-digit numbers.

A two-way table would be a systematic way to list all the outcomes

	2	3	5	7
1	12	13	15	17
4	42	43	45	47
9	92	93	95	97

b) Find $\mathrm{P}(T)$.

$\mathrm{P}(T) = \dfrac{n(T)}{n(U)}$ ← Number of multiples of 3 / ← Total number of outcomes

$\{12, 15, 42, 45, 93\}$ are the multiples of 3

$\boxed{\mathrm{P}(T) = \dfrac{5}{12}}$

c) Find $\mathrm{P}(T')$.

$\mathrm{P}(T) + \mathrm{P}(T') = 1 \Rightarrow \mathrm{P}(T') = 1 - \mathrm{P}(T)$

$\mathrm{P}(T') = 1 - \dfrac{5}{12}$

$\boxed{\mathrm{P}(T') = \dfrac{7}{12}}$

Independent & Mutually Exclusive Events

What are mutually exclusive events?

- Two events are **mutually exclusive** if they **cannot both occur**
  - For example: when rolling a dice the events "getting a prime number" and "getting a 6" are mutually exclusive
- If *A* and *B* are mutually exclusive events then:
  - $\mathrm{P}(A \cap B) = 0$

What are independent events?

- Two events are **independent** if **one occurring does not affect the probability of the other occurring**
  - For example: when flipping a coin twice the events “getting a tails on the first flip” and “getting a tails on the second flip” are independent
- If *A* and *B* are independent events then:
  - $\mathrm{P}(A | B) = \mathrm{P}(A)$ and $\mathrm{P}(B|A) = \mathrm{P}(B)$
- If *A* and *B* are independent events then:
  - $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
    - This is given in the **formula booklet**
    - This is a useful formula to test whether two events are statistically independent

How do I find the probability of combined mutually exclusive events?

- If *A* and *B* are **mutually exclusive** events then

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

    - This is given in the **formula booklet**
    - This occurs because $\mathrm{P}(A \cap B) = 0$
- For any two events *A* and *B* the events $A \cap B$ and $A \cap B'$ are **mutually exclusive** and *A* is the **union** of these two events
  - $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B')$
    - This works for any two events *A* and *B*

Worked example

a) A student is chosen at random from a class. The probability that they have a dog is 0.8, the probability they have a cat is 0.6 and the probability that they have a cat or a dog is 0.9. Find the probability that the student has both a dog and a cat.

Let D be event "has a dog" and C be "has a cat"

$P(D \cup C) = P(D) + P(C) - P(D \cap C)$

$0.9 = 0.8 + 0.6 - P(D \cap C)$

$\boxed{P(D \cap C) = 0.5}$

b) Two events, Q and R, are such that $\mathrm{P}(Q) = 0.8$ and $\mathrm{P}(Q \cap R) = 0.1$. Given that Q and R are independent, find $\mathrm{P}(R)$.

Q and R independent $\Rightarrow P(Q \cap R) = P(Q)P(R)$

$0.1 = 0.8 \times P(R) \quad \therefore \quad P(R) = \frac{0.1}{0.8}$

$\boxed{P(R) = 0.125 \text{ or } \frac{1}{8}}$

c) Two events, S and T, are such that $\mathrm{P}(S) = 2\mathrm{P}(T)$. Given that S and T are mutually exclusive and that $\mathrm{P}(S \cup T) = 0.6$ find $\mathrm{P}(S)$ and $\mathrm{P}(T)$.

S and T mutually exclusive $\Rightarrow P(S \cup T) = P(S) + P(T)$

$0.6 = P(S) + P(T)$

$0.6 = 2P(T) + P(T)$ $\quad P(S) = 2P(T)$

$0.6 = 3P(T)$

$\boxed{P(T) = 0.2 \text{ and } P(S) = 0.4}$

4.3.2 Conditional Probability

Conditional Probability

What is conditional probability?

- **Conditional probability** is where the probability of an **event** happening can vary depending on the outcome of a prior event
- The event A happening **given that** event B has happened is denoted $A|B$
- A common example of conditional probability involves selecting multiple objects from a bag **without replacement**
  - The probability of selecting a certain item changes depending on what was selected before
    - This is because the total number of items will change as they are not replaced once they have been selected

How do I calculate conditional probabilities?

- Some conditional probabilities can be calculated by using counting outcomes
  - Probabilities without replacement can be calculated like this
  - For example: There are 10 balls in a bag, 6 of them are red, two of them are selected without replacement
    - To find the probability that the second ball selected is red given that the first one is red count how many balls are left:
    - A red one has already been selected so there are 9 balls left and 5 are red so the probability is $\frac{5}{9}$
- You can use sample space diagrams to find the probability of A given B:
  - reduce your sample space to just include outcomes for event B
  - find the proportion that also contains outcomes for event A
- There is a formula for conditional probability that you should use
  - $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
  - This is given in the **formula booklet**
  - This can be rearranged to give $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A|B)$
  - By symmetry you can also write $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A)$

How do I tell if two events are independent using conditional probabilities?

- If A and B are two events then they are independent if:
  - $\mathrm{P}(A|B) = \mathrm{P}(A) = \mathrm{P}(A|B')$
- Equally you can still use $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ to test for independence
  - This is given in the **formula booklet**

Worked example

Let R be the event that it is raining in Weatherville and T be the event that there is a thunderstorm in Weatherville.

It is known that $\mathrm{P}(T) = 0.035$, $\mathrm{P}(T \cap R) = 0.03$ and $\mathrm{P}(T \mid R) = 0.15$.

a) Find the probability that it is raining in Weatherville.

Formula booklet

Conditional probability	$\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

$$P(T \mid R) = \frac{P(T \cap R)}{P(R)}$$

Substitute the values in

$$0.15 = \frac{0.03}{P(R)}$$

$$P(R) = \frac{0.03}{0.15}$$

$$\boxed{P(R) = 0.2}$$

b) State whether the events R and T are independent. Give a reason for your answer.

If R and T are independent then $P(T \mid R) = P(T)$

$P(T \mid R) = 0.15$ and $P(T) = 0.035$

R and T are not independent as $P(T \mid R) \neq P(T)$

4.3.3 Bayes' Theorem

Bayes' Theorem

What is Bayes' Theorem

- **Bayes' Theorem** allows you **switch the order** of conditional probabilities
  - If you know $\mathrm{P}(B)$, $\mathrm{P}(B')$ and $\mathrm{P}(A \mid B)$ then Bayes' Theorem allows you to find $\mathrm{P}(B \mid A)$
- Essentially if you have a **tree diagram** you will already know the conditional probabilities of the **second branches**
  - Bayes' Theorem allows you to find the **conditional probabilities** if you **switch the order** of the events
- For any two events *A* and *B* Bayes' Theorem states:

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(B)\mathrm{P}(A \mid B)}{\mathrm{P}(B)\mathrm{P}(A \mid B) + \mathrm{P}(B')\mathrm{P}(A \mid B')}$$

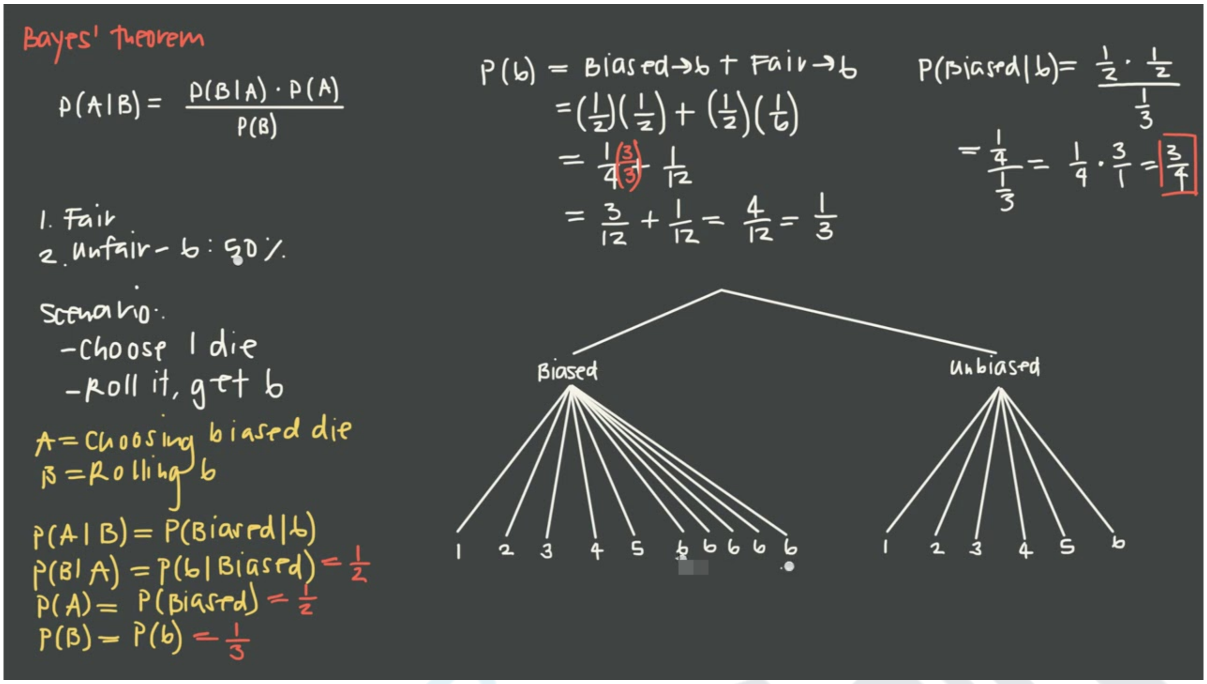
  - This is given in the **formula booklet**
  - This formula is derived using the formulae:
    - $\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$
    - $\mathrm{P}(A) = \mathrm{P}(B \cap A) + \mathrm{P}(B' \cap A)$
    - $\mathrm{P}(B \cap A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ and $\mathrm{P}(B' \cap A) = \mathrm{P}(B')\mathrm{P}(A \mid B')$
- Bayes' Theorem can be **extended** to **mutually exclusive events** $B_1, B_2, \ldots, B_n$ and any other event *A*
  - In your exam you will have a **maximum of three** mutually exclusive events

$$\mathrm{P}(B_i \mid A) = \frac{\mathrm{P}(B_i)\mathrm{P}(A \mid B_i)}{\mathrm{P}(B_1)\mathrm{P}(A \mid B_1) + \mathrm{P}(B_2)\mathrm{P}(A \mid B_2) + \mathrm{P}(B_3)\mathrm{P}(A \mid B_3)}$$

    - This is given in the **formula booklet**

How do I calculate conditional probabilities using Bayes' Theorem?

- Start by **drawing a tree diagram**
  - Label B_1 & B_2 (& B_3 if necessary) on the **first set** of branches
  - Label ***A* & *A'*** on the **second set** of branches
- The questions will give you enough information to **label the probabilities** on this tree
- **Identify the probabilities needed** to use Bayes' Theorem
  - The probabilities will come in pairs: $\mathrm{P}(B_i)$ and $\mathrm{P}(A \mid B_i)$

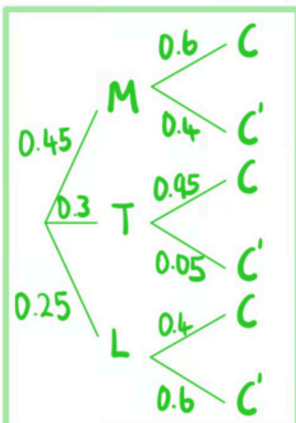

Bayes' theorem
$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$
1. Fair
2. unfair - 6 : 50%
Scenario:
-choose 1 die
-Roll it, get 6
A = choosing biased die
B = Rolling 6
$P(A|B) = P(\text{Biased}|6)$
$P(B|A) = P(6|\text{Biased}) = \frac{1}{2}$
$P(A) = P(\text{Biased}) = \frac{1}{2}$
$P(B) = P(6) = \frac{1}{3}$
$P(6) = \text{Biased} \to 6 + \text{Fair} \to 6$
$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)$
$= \frac{1}{4}\left(\frac{3}{3}\right) + \frac{1}{12}$
$= \frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$
$P(\text{Biased}|6) = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{3}}$
$= \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{1}{4} \cdot \frac{3}{1} = \boxed{\frac{3}{4}}$
Biased
Unbiased
1 2 3 4 5 6 6 6 6 6
1 2 3 4 5 6

Worked example

Lucy is doing a quiz. For each question there's a 45% chance that it is about music, 30% chance that it is about TV and 25% chance that it is about literature. The probability that Lucy answers a question correctly is 0.6 for music, 0.95 for TV and 0.4 for literature.

a) Draw a tree diagram to represent this information.

Let M be the event that the question is on music
T be the event that the question is on TV
L be the event that the question is on literature
C be the event that Lucy answers correctly



$P(C'|M) = 1 - P(C|M)$
$= 1 - 0.6$

$P(C'|T) = 1 - P(C|T)$
$= 1 - 0.95$

$P(C'|L) = 1 - P(C|L)$
$= 1 - 0.4$

b) Given that Lucy answered a question correctly, find the probability that it was about TV.

Formula booklet

Bayes' theorem	$P(B_i \mid A) = \dfrac{P(B_i)P(A \mid B_i)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)}$

Let $A = C$ $\quad B_1 = M$ $\quad B_2 = T$ $\quad B_3 = L$

$$P(T|C) = \frac{P(T)P(C|T)}{P(M)P(C|M) + P(T)P(C|T) + P(L)P(C|L)}$$

$$= \frac{0.3 \times 0.95}{0.45 \times 0.6 + 0.3 \times 0.95 + 0.25 \times 0.4}$$

$$\boxed{P(T|C) = \frac{57}{131} = 0.435 \ (3sf)}$$

4.3.4 Sample Space Diagrams

Venn Diagrams

What is a Venn diagram?

- A Venn diagram is a way to illustrate **events** from an **experiment** and are particularly useful when there is an overlap between possible **outcomes**
- A Venn diagram consists of
  - a **rectangle** representing the **sample space (*U*)**
    - The rectangle is labelled *U*
    - Some mathematicians instead use *S* or *ξ*
  - a **circle** for each **event**
    - Circles may or may not overlap depending on which **outcomes** are shared between **events**
- The numbers in the circles represent either the **frequency** of that event or the **probability** of that event
  - If the **frequencies** are used then they should **add up to the total frequency**
  - If the **probabilities** are used then they should **add up to 1**

What do the different regions mean on a Venn diagram?

- A' is represented by the regions that are **not in** the *A* circle
- $A \cap B$ is represented by the region where the *A* and *B* circles **overlap**
- $A \cup B$ is represented by the regions that **are in** *A* or *B* or both
- Venn diagrams show '**AND**' and '**OR**' statements easily
- Venn diagrams also instantly show **mutually exclusive** events as these circles will **not overlap**
- **Independent** events can not be instantly seen
  - You need to use probabilities to deduce if two events are independent

How do I solve probability problems involving Venn diagrams?

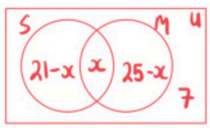
- Draw, or add to a given Venn diagram, filling in as many values as possible from the information provided in the question
- It is usually helpful to work from the centre outwards
  - Fill in **intersections** (overlaps) first
- If two events are independent you can use the formula
  - $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$
- To find the conditional probability $\mathrm{P}(A \mid B)$
  - Add together the frequencies/probabilities in the *B* circle
    - This is your denominator
  - Out of those frequencies/probabilities add together the ones that are also in the *A* circle
    - This is your numerator
  - Evaluate the fraction

Worked example

40 people are asked if they have sugar and/or milk in their coffee. 21 people have sugar, 25 people have milk and 7 people have neither.

a) Draw a Venn diagram to represent the information.

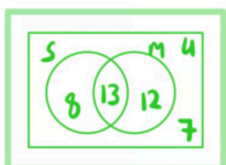
Find the centre first



Total should be 40

$(21-x)+x+(25-x)+7=40$

$53-x=40 \quad \therefore\ x=13$



b) One of the 40 people are randomly selected, find the probability that they have sugar but not milk with their coffee.

S and not M is the part of S circle that does not include M

$P(S \cap M') = \frac{8}{40}$

Remember to write as a fraction of the total

$\boxed{P(S \cap M') = \frac{1}{5}}$

c) Given that a person who has sugar is selected at random, find the probability that they have milk with their coffee.

Given that sugar has been selected we only want the S circle as our total.

Out of the S circle 13 also have milk

$\boxed{P(M|S) = \frac{13}{21}}$

Tree Diagrams

What is a tree diagram?

- A **tree diagram** is another way to show the outcomes of combined events
  - They are very useful for intersections of events
- The events on the branches must be **mutually exclusive**
  - Usually they are an event and its complement
- The probabilities on the second sets of branches **can depend** on the outcome of the first event
  - These are **conditional probabilities**
- When selecting the items from a bag:
  - The second set of branches will be the **same** as the first if the items **are replaced**
  - The second set of branches will be the **different** to the first if the items **are not replaced**

How are probabilities calculated using a tree diagram?

- To find the probability that two events happen together you **multiply** the corresponding probabilities on their branches
  - It is helpful to find the probability of all combined outcomes once you have drawn the tree
- To find the probability of an event you can:
  - **add together** the probabilities of the **combined outcomes** that are part of that event
    - For example: $\mathrm{P}(A \cup B) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B)$
  - **subtract** the probabilities of the combined outcomes that are not part of that event from 1
    - For example: $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(A' \cap B')$

Do I have to use a tree diagram?

- If there are **multiple events** or trials then a tree diagram can get big
- You can break down the problem by using the words **AND/OR/NOT** to help you find probabilities without a tree
- You can speed up the process by only drawing parts of the tree that you are interested in

Which events do I put on the first branch?

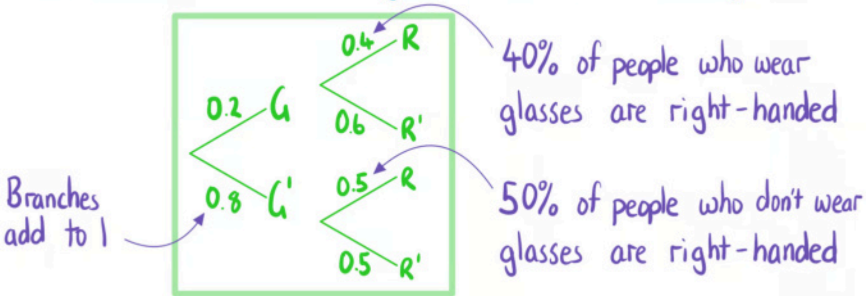
- If the events *A* and *B* are **independent** then the **order does not matter**
- If the events *A* and *B* are **not independent** then the **order does matter**
  - If you have the probability of ***A* given *B*** then put ***B* on the first set** of branches
  - If you have the probability of ***B* given *A*** then put ***A* on the first set** of branches

Worked example

20% of people in a company wear glasses. 40% of people in the company who wear glasses are right-handed. 50% of people in the company who don't wear glasses are right-handed.

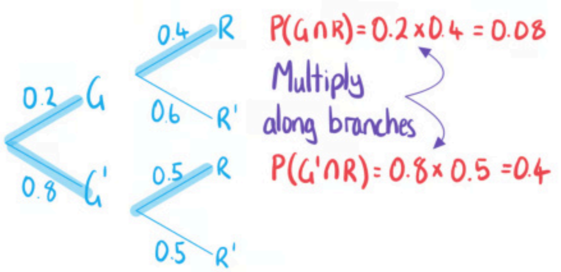
a) Draw a tree diagram to represent the information.

Let G be the event "wears glasses" and R be "is right-handed"



b) One of the people in the company are randomly selected, find the probability that they are right-handed.

Find options that contain R



$P(R) = P(G \cap R) + P(G \cap R') = 0.08 + 0.4$

$\boxed{P(R) = 0.48}$

c) Given that a person who is right-handed is selected at random, find the probability that they wear glasses.

$$P(G|R) = \frac{P(G \cap R)}{P(R)} = \frac{0.08}{0.48}$$

$$\boxed{P(G|R) = \frac{1}{6}}$$