DP IB Maths: AI HL

4.13 Transition Matrices & Markov Chains

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4.13.1 Markov Chains

Markov Chains

What is meant by a "state"?

- States refer to **mutually exclusive events** with the current event **able to change over time**
- Examples of states include:
  - Daily weather conditions
    - The states could be: "sunny" and "not sunny"
  - Countries visited by an inspector each day
    - The states could be: "France", "Spain" and "Germany"
  - Store chosen for weekly grocery shop:
    - The states could be: "Foods-U-Like", "Smiley Shoppers" and "Better Buys"

What is a Markov chain?

- A **Markov chain** is a model that describes a **sequence of states** over a period of time
  - Time is measured in discrete steps
    - Such as days, months, years, etc
- The **conditions** for a Markov chain are:
  - The **probability** of a state being the **next state** in the sequence **only depends** on the **current state**
    - For example
      The 11^{th} state **only depends** on the 10^{th} state
      The first 9 states **do not affect** the 11^{th} state
    - This probability is called a **transition probability**
  - The **transition probabilities do not change** over time
    - For example
      The probability that the 11^{th} state is A given that the 10^{th} state is B is equal to the probability that the 12^{th} state is A given that the 11^{th} state is B
- A Markov chain is said to be **regular** if there is a value *k* such that in **exactly *k* steps** it is possible to reach any state regardless of the initial state
  - The chain where A can only go to B, B can only go to C and C can only go to A, is **not regular**
    - After any number of changes, A can only go to either B or C but not both
    - After 100 changes, A can end up at B but not C
    - After 500 changes, A can end up at C but not B

What is a transition state diagram?

- A **transition diagram** is a **directed graph**
  - The **vertices** are the **states**
  - The **edges** represent the **transition probabilities** between the states
- The graph can contain
  - **Loops**
    - These will be the transition probabilities of the next state being the same as the current state

   - **Two edges between each pair** of vertices
     - The edges will be in opposite directions
     - Each edge will show the transition probability of the state changing in the given direction
- The **probabilities** on the **edges coming out** of a vertex **add up to 1**

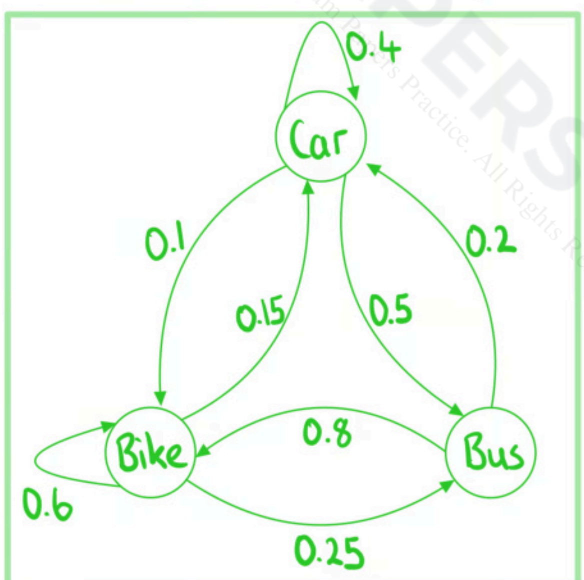
Worked example

Fleur travels to work by car, bike or bus. Each day she chooses her mode of transport based on the transport she chose the previous day.

- If Fleur travels by car then there is a 40% chance that she will travel by car the following day and a 10% chance that she will travel by bike.
- If Fleur travels by bike then there is a 60% chance that she will travel by bike the following day and a 25% chance that she will travel by bus.
- If Fleur travels by bus then there is an 80% chance that she will travel by bike the following day and a 20% chance that she will travel by car.

Represent this information as a transition state diagram.

The probabilities on the arrows coming out of a state add to 1



4.13.2 Transition Matrices

Transition Matrices

What is a transition matrix?

- A **transition matrix *T*** shows the **transition probabilities** between the current state and the next state
  - The **columns** represent the **current states**
  - The **rows** represent the **next states**
- The element of ***T*** in the i^{th} row and j^{th} column gives the transition probability t_{ij} of :
  - the **next state** being the state corresponding to **row *i***
  - **given that the current state** is the state corresponding to **column *j***
- The probabilities in each **column** must **add up to 1**
- The transition matrix depends on how you assign the states to the columns
  - Each transition matrix for a Markov chain will contain the same elements
    - The rows and columns may be in different orders though
    - E.g. Sunny (S) & Cloudy (C) could be in the order **S then C** or **C then S**

What is an initial state probability matrix?

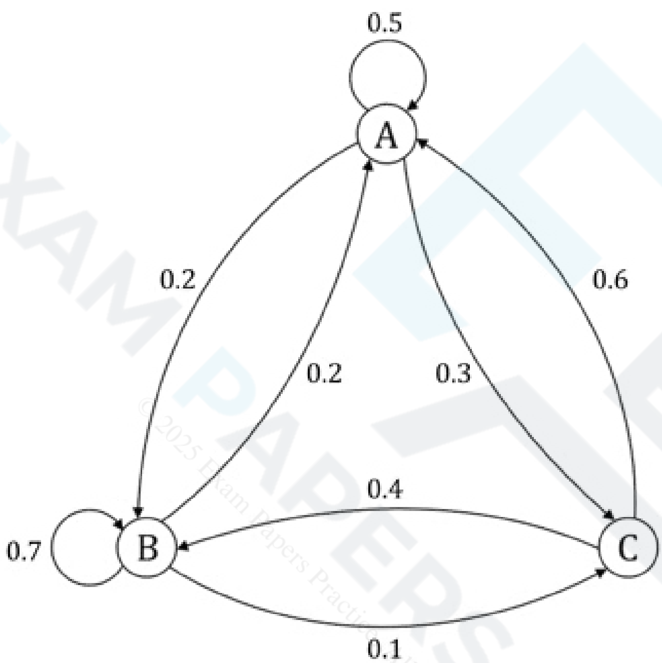
- An **initial state probability matrix s_0** is a column vector which contains the **probabilities** of each state being chosen as the **initial state**
  - If you know which state was chosen as the initial state then that entry will be 1 and the others will all be zero
- You can find the **state probability matrix s_1** which contains the probabilities of each state being chosen after **one interval of time**
  - $s_1 = Ts_0$

How do I find expected values after one interval of time?

- Suppose the Markov change represents a **population moving between states**
  - Examples include:
    - People in a town switching gyms each year
    - Children choosing a type of sandwich for their lunch each day
- Suppose the **total population is fixed** and equals N
- You can **multiply the state probability matrix s_1** by N to find the expected number of members of the population at each state

Worked example

Each year Jamie donates to one of three charities: A, B or C. At the start of each year, the probabilities of Jamie continuing donate to the same charity or changing charities are represented by the following transition state diagram:



a) Write down a transition matrix $\boldsymbol{T}$ for this system of probabilities.

$$
\begin{array}{cc}
 & \text{Current state} \\
 & \begin{array}{ccc} A & B & C \end{array} \\
\text{Next state} \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0.5 & 0.2 & 0.6 \\ 0.2 & 0.7 & 0.4 \\ 0.3 & 0.1 & 0 \end{pmatrix}
\end{array}
\qquad
\boxed{T = \begin{pmatrix} 0.5 & 0.2 & 0.6 \\ 0.2 & 0.7 & 0.4 \\ 0.3 & 0.1 & 0 \end{pmatrix}}
$$

b) There is a 10% chance that charity A is the first charity that Jamie chooses, a 10% chance for charity B and an 80% chance for charity C. Find the charity which has the highest probability of being picked as the second charity after the first year.

Write down the initial state vector $s_0 = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.8 \end{pmatrix}$

$s_1 = Ts_0$ $s_1 = \begin{pmatrix} 0.5 & 0.2 & 0.6 \\ 0.2 & 0.7 & 0.4 \\ 0.3 & 0.1 & 0 \end{pmatrix}\begin{pmatrix} 0.1 \\ 0.1 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.55 \\ 0.41 \\ 0.04 \end{pmatrix}$

Charity A has the highest probability of being the second charity picked.

Powers of Transition Matrices

How do I find powers of a transition matrix?

- You can simply use your **GDC** to find given powers of a matrix
- The power could be left in terms of an **unknown** n
  - In this case it would be more helpful to write the transition matrix in diagonalised form (see section **1.8.2 Applications of Matrices**) $T = PDP^{-1}$ where
    - D is a **diagonal matrix** of the **eigenvalues**
    - P is a matrix of **corresponding eigenvectors**
  - Then $T^n = PD^nP^{-1}$
    - This is given in the **formula booklet**
  - Every transition matrix always has an **eigenvalue equal to 1**

What is represented by the powers of a transition matrix?

- The powers of a transition matrix also **represent probabilities**
- The element of T^n in the i^{th} row and j^{th} column gives the **probability** t^n_{ij} of :
  - the **future state** after **n intervals of time** being the state corresponding to **row i**
  - **given that** the **current state** is the state corresponding to **column j**
- For example: Let T be a transition matrix with the element $t_{2,3}$ representing the probability that tomorrow is sunny given that it is raining today
  - The element $t^5_{2,3}$ of the matrix T^5 represents the probability that it is sunny in 5 days' time given that it is raining today
- The probabilities in **each column** must still **add up to 1**

How do I find the column state matrices?

- The column state matrix $\mathbf{s_n}$ is a column vector which contains the **probabilities** of each state being chosen after n intervals of time given the current state
  - $\mathbf{s_n}$ depends on $\mathbf{s_0}$
- To calculate the column state matrix you raise the transition matrix to the power n and multiply by the initial state matrix
  - $\boldsymbol{T}^n \boldsymbol{s}_0 = \boldsymbol{s}_n$
    - You are given this in the **formula booklet**
- You can multiply $\mathbf{s_n}$ by the fixed population size to find the expected number of members of the population at each state after n intervals of time

Worked example

At a cat sanctuary there are 1000 cats. If a cat is brushed on a given day, then the probability it is brushed the following day is 0.2. If a cat is not brushed on a given day, then the probability that is will be brushed the following day is 0.9.

The transition matrix $\boldsymbol{T}$ is used to model this information with $\boldsymbol{T}=\begin{pmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{pmatrix}$.

a) On Monday Hippo the cat is brushed. Find the probability that Hippo will be brushed on Friday.

Identify the states with the rows/columns

$$\begin{array}{cc} & \text{Current} \\ & \begin{array}{cc} B & B' \end{array} \\ \text{Next} \begin{array}{c} B \\ B' \end{array} & \begin{pmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{pmatrix} \end{array}$$

Friday is 4 days after Monday

$$T^4 = \begin{pmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{pmatrix}^4 = \underbrace{\begin{pmatrix} 0.6424 & 0.4023 \\ 0.3576 & 0.5977 \end{pmatrix}}_{\begin{array}{cc} B & B' \end{array} \text{ Current}} \left.\begin{array}{c} B \\ B' \end{array}\right\} \text{Future}$$

Current = B

Future = B

$\boxed{0.6424}$

b) On Monday 700 cats were brushed. Find the expected number of cats that will be brushed on the following Monday.

On Monday 700 brushed $\quad S_0 = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

Expected numbers after 7 days

$\text{Total} \times S_7 = \text{Total} \times T^7 S_0$

$$1000 \times \begin{pmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{pmatrix}^7 \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{pmatrix}^7 \begin{pmatrix} 700 \\ 300 \end{pmatrix} = \begin{pmatrix} 515.36309 \\ 484.63691 \end{pmatrix} \begin{matrix} B \\ B' \end{matrix}$$

$\boxed{515 \text{ cats}}$

Steady State & Long-term Probabilities

What is the steady state of a regular Markov chain?

- The vector **s** is said to be a **steady state** vector if it does not change when multiplied by the transition matrix
  - $\boldsymbol{T}\mathbf{s} = \mathbf{s}$
- **Regular Markov chains** have steady states
  - A Markov chain is said to be regular if there exists a **positive integer *k*** such that **none of the entries** are **equal to 0** in the matrix $\boldsymbol{T}^k$
    - For this course all Markov chains will be regular
- The transition matrix for a regular Markov chain will have **exactly one** eigenvalue equal to 1 and the **rest will all be less than 1**
- As n gets bigger $\boldsymbol{T}^n$ tends to a matrix where **each column is identical**
  - The column matrix formed by using **one of these columns** is called the steady state column matrix **s**
  - This means that the **long-term probabilities** tend to fixed probabilities
    - $\mathbf{s}_n$ tends to **s**

How do I use long-term probabilities to find the steady state?

- As $\boldsymbol{T}^n$ tends to a matrix whose columns equal the steady state vector
  - Calculate $\boldsymbol{T}^n$ for a large value of n using your GDC
  - If the columns are identical when rounded to a required degree of accuracy then the column is the steady state vector
  - If the columns are not identical then choose a higher power and repeat

How do I find the exact steady state probabilities?

- As $\boldsymbol{T}\mathbf{s} = \mathbf{s}$ the steady state vector **s** is the **eigenvector** of ***T*** corresponding to the **eigenvalue equal to 1** whose elements sum to 1:
  - Let **s** have entries $x_1, x_2, \ldots, x_n$
  - Use $\boldsymbol{T}\mathbf{s} = \mathbf{s}$ to form a system of linear equations
  - There will be an infinite number of solutions so choose a value for one of the unknowns
    - For example: let $x_n = 1$
  - Ignoring the last equation solve the system of linear equations to find $x_1, x_2, \ldots, x_{n-1}$
  - Divide each value x_i by the sum of the values
    - This makes the values add up to 1
- You might be asked to **show this result using diagonalisation**
  - Write $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$ where ***D*** is the diagonal matrix of eigenvalues and ***P*** is the matrix of eigenvectors
  - Use $\boldsymbol{T}^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}$
  - As n gets large $\boldsymbol{D}^n$ tends to a matrix where all entries are 0 apart from one entry of 1 due to the eigenvalue of 1
  - Calculate the limit of $\boldsymbol{T}^n$ which will have **identical columns**
    - You can calculate this by multiplying the three matrices ($\boldsymbol{P}, \boldsymbol{D}^{\infty}, \boldsymbol{P}^{-1}$) together

Worked example

If a cat is brushed on a given day, then the probability it is brushed the following day is 0.2. If a cat is not brushed on a given day, then the probability that is will be brushed the following day is 0.9.

The transition matrix $\boldsymbol{T}$ is used to model this information with $\boldsymbol{T} = \begin{pmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{pmatrix}$.

a) Find an eigenvector of $\boldsymbol{T}$ corresponding to the eigenvalue 1.

$\underline{v}$ is an eigenvector of T with eigenvalue 1 if $T\underline{v} = \underline{v}$

Let $\underline{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$T\underline{v} = \begin{pmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.2x_1 + 0.9x_2 \\ 0.8x_1 + 0.1x_2 \end{pmatrix}$

$T\underline{v} = \underline{v}$ $\quad 0.2x_1 + 0.9x_2 = x_1 \Rightarrow 0.9x_2 = 0.8x_1 \Rightarrow 9x_2 = 8x_1$

$\quad 0.8x_1 + 0.1x_2 = x_2 \Rightarrow 0.8x_1 = 0.9x_2 \Rightarrow 8x_1 = 9x_2$

Find a solution $\quad x_1 = 9$ and $x_2 = 8$

$\boxed{\begin{pmatrix} 9 \\ 8 \end{pmatrix} \text{ or any scalar multiple}}$

b) Hence find the steady state vector.

Scale the elements so that they add to 1 $\begin{pmatrix} \frac{9}{17} \\ \frac{8}{17} \end{pmatrix}$

The eigenvector corresponding to the eigenvalue 1, whose elements add to 1, is the steady state vector.

$$\boxed{\begin{pmatrix} \frac{9}{17} \\ \frac{8}{17} \end{pmatrix}}$$