DP IB Maths: AA HL

3.9 Vector Properties

Contents

3.9.1 Introduction to Vectors

Scalars & Vectors

What are scalars?

- **Scalars** are quantities without **direction**
  - They have only a **size (magnitude)**
  - For example: **speed, distance, time, mass**
- **Most scalar quantities** can never be **negative**
  - You cannot have a negative speed or distance

What are vectors?

- **Vectors** are quantities which also have a **direction**, this is what makes them more than just a scalar
  - For example: two objects with **velocities** of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- A **vector quantity** is described by both its **magnitude** and **direction**
- A vector has **components** in the direction of the *x*- , *y*-, and *z*- axes
  - Vector quantities can have **positive** or **negative** components
- Some examples of vector quantities you may come across are **displacement**, **velocity**, **acceleration**, **force/weight**, **momentum**
  - **Displacement** is the position of an object from a starting point
  - **Velocity** is a speed in a given direction (displacement over time)
  - **Acceleration** is the change in velocity over time
- Vectors may be given in either 2– or 3– dimensions

Worked example

State whether each of the following is a scalar or a vector quantity.

a) A speed boat travels at 3 m/s on a bearing of 052°

Speed with a given direction → velocity

$\boxed{\text{Vector}}$

b) A garden is 1.7 m wide

Length with no direction

$\boxed{\text{Scalar}}$

c) A car accelerates forwards at 5.4 ms^{-2}

Acceleration has direction

$\boxed{\text{Vector}}$

d) A film lasts 2 hours 17 minutes

Time has no direction

$\boxed{\text{Scalar}}$

e) An athlete runs at an average speed of 10.44 ms^{-1}

Speed with no direction is a scalar

$\boxed{\text{Scalar}}$

f) A ball rolls forwards 60 cm before stopping

Displacement has direction

$\boxed{\text{Vector}}$

Vector Notation

How are vectors represented?

- **Vectors** are usually represented using an arrow in the direction of movement
  - The length of the arrow represents its magnitude
- They are written as lowercase letters either in **bold** or underlined
  - For example a vector from the point O to A will be written **a** or $\underline{a}$
    - The vector from the point A to O will be written **–a** or -$\underline{a}$
- If the start and end point of the vector is known, it is written using these points as capital letters with an arrow showing the direction of movement
  - For example: $\overrightarrow{AB}$ or $\overrightarrow{BA}$
- Two vectors are equal only if their corresponding components are equal
- Numerically, vectors are either represented using **column vectors** or **base vectors**
  - Unless otherwise indicated, you may carry out all working and write your answers in either of these two types of vector notation

What are column vectors?

- **Column vectors** are where one number is written above the other enclosed in brackets
- In 2–dimensions the **top number** represents movement in the **horizontal direction** (right/left) and the **bottom number** represents movement in the **vertical direction** (up/down)
- A **positive value** represents movement in the **positive direction** (right/up) and a **negative value** represents movement in the **negative direction** (left/down)
  - For example: The column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ represents **3 units** in the **positive horizontal** (*x*) direction (i.e., **right**) and **2 units** in the **negative vertical** (*y*) direction (i.e., **down**)
- In 3–dimensions the **top number** represents the movement in the ***x* direction** (length), the **middle number** represents movement in the ***y* direction** (width) and the **bottom number** represents the movement in the ***z* direction** (depth)
  - For example: The column vector $\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$ represents **3 units** in the **positive *x* direction**, **4 units** in the **negative *y* direction** and **2 units** in the **positive *z* direction**

What are base vectors?

- **Base vectors use i**, **j** and **k** notation **where i**, **j** and **k** are **unit vectors** in the positive *x*, *y*, and *z* directions respectively
  - This is sometimes also called **unit vector notation**
  - A unit vector has a **magnitude of 1**
- In 2–dimensions i represents movement in the horizontal direction (right/left) and j represents the movement in the vertical direction (up/down)

- **For example:** The vector (-4**i** + 3**j**) would mean **4 units** in the **negative horizontal** (x) direction (i.e., **left**) and **3 units** in the **positive vertical** (y) direction (i.e., **up**)
- **I**n 3–dimensions i represents movement in the x direction (length), j represents movement in the y direction (width) and k represents the movement in the z direction (depth)
  - **For example:** The vector (-4**i** + 3**j – k**) would mean **4 units** in the **negative *x* direction**, **3 units** in the **positive *y* direction** and **1 unit** in the **negative *z* direction**
- As they are vectors, **i, j and k** are displayed in **bold** in textbooks and online but in handwriting they would be underlined (i, j and k)

Worked example

a) Write the vector $\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$ using **base vector** notation.

$$\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} = -4\underline{i} + 0\underline{j} + 5\underline{k}$$

0j is not needed when giving answer in base vector form.

$$\boxed{5\underline{k} - 4\underline{i}}$$

b) Write the vector $\mathbf{k} - 2\mathbf{j}$ using **column vector** notation.

$$\underline{k} - 2\underline{j} = 0\underline{i} - 2\underline{j} + 1\underline{k}$$

Be careful with negative components and missing terms when working with base vectors

$$\boxed{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}$$

The zero term is needed when using column vector notation

Parallel Vectors

How do you know if two vectors are parallel?

- Two vectors are parallel if one is a **scalar multiple** of the other
  - This means that all components of the vector have been multiplied by a **common constant (scalar)**
- Multiplying every component in a vector by a **scalar** will change the **magnitude** of the vector but not the **direction**
  - For example: the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{b} = 2\mathbf{a} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$ will have the **same direction** but the vector **b** will have twice the magnitude of **a**
    - They are **parallel**
- If a vector can be factorised by a **scalar** then it is parallel to any **scalar multiple** of the factorised vector
  - For example: The vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ can be factorised by the scalar 3 to $3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ so the vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ is parallel to any **scalar multiple** of $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- If a vector is multiplied by a **negative scalar** its direction will be **reversed**
  - It will still be **parallel** to the original vector
- Two vectors are **parallel** if they have the same or reverse **direction** and **equal** if they have the same **size and direction**

Worked example

Show that the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{b} = 6\mathbf{k} - 3\mathbf{i}$ are parallel and find the scalar multiple that maps **a** onto **b**.

Convert both vectors into the same form and then look for a value of k such that $\underline{a} = k\underline{b}$, where k is a scalar.

$\underline{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$\underline{b} = 6\underline{k} - 3\underline{i} = -3\underline{i} + 0\underline{j} + 6\underline{k}$

$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$= -\frac{3}{2}\underline{a}$

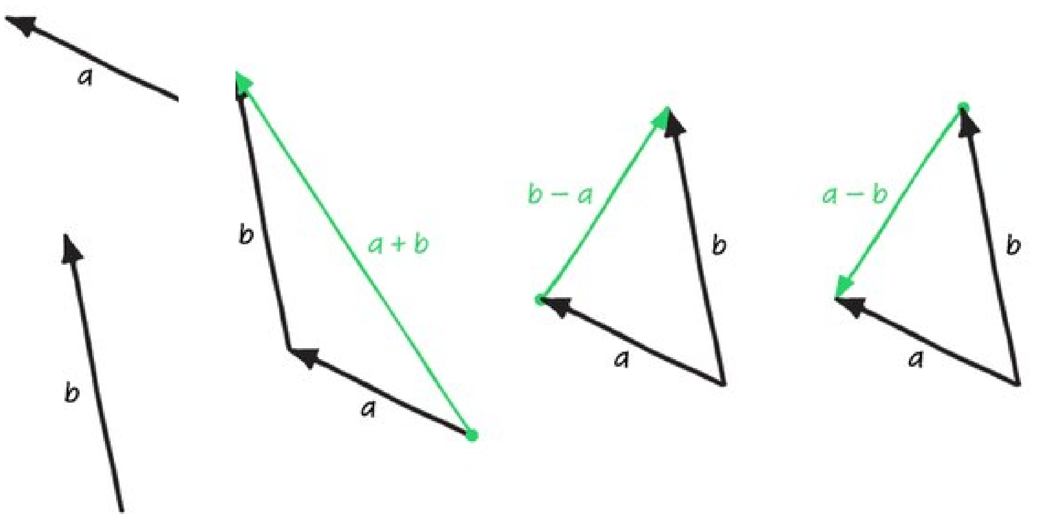
$\underline{b} = -\frac{3}{2}\underline{a}$, $k = -\frac{3}{2}$

3.9.2 Position & Displacement Vectors

Adding & Subtracting Vectors

How are vectors added and subtracted numerically?

- To **add** or **subtract** vectors numerically simply add or subtract each of the corresponding components
- In **column vector** notation just add the top, middle and bottom parts together
  - For example: $\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -8 \end{pmatrix}$
- In **base vector** notation add each of the **i**, **j**, and **k** components together separately
  - For example: (2**i** + **j** – 5**k**) – (**i** + 4**j** + 3**k**) = (**i** – 3**j** – 8**k**)



How are vectors added and subtracted geometrically?

- Vectors can be **added** geometrically by joining the end of one vector to the start of the next one
- The **resultant** vector will be the shortest route from the start of the first vector to the end of the second
  - A **resultant** vector is a vector that results from **adding** or **subtracting** two or more vectors
- If the two vectors have the same **starting position**, the second vector can be **translated** to the end of the first vector to find the resultant vector
  - This results in a **parallelogram** with the resultant vector as the diagonal
- To **subtract** vectors, consider this as **adding on the negative vector**
  - For example: **a** – **b** = **a** + (–**b**)
  - The end of the **resultant vector a – b** will not be anywhere near the end of the vector **b**
    - Instead, it will be at the point where the end of the vector **–b** would be

Worked example

Find the resultant of the vectors **a** = 5**i** – 2**j** and **b** = $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

$\underline{a} = 5\underline{i} - 2\underline{j} + 0\underline{k} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

Writing as a column vector makes adding and subtracting easier.

$\underline{a} + \underline{b} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\boxed{\text{Resultant vector} = 2\underline{i} - \underline{j} + 2\underline{k}}$

Position Vectors

What is a position vector?

- A position vector describes the **position** of a point in relation to the **origin**
  - It describes the **direction** and the **distance** from the point O: 0**i** + 0**j** + 0**k** or $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
  - It is different to a **displacement vector** which describes the direction and distance between any two points
- The position vector of point A is written with the notation **a** = $\overrightarrow{OA}$
  - The origin is always denoted O
- The individual components of a position vector are the coordinates of its end point
  - For example the point with coordinates (3, -2, -1) has position vector 3**i** – 2**j** – **k**

Worked example

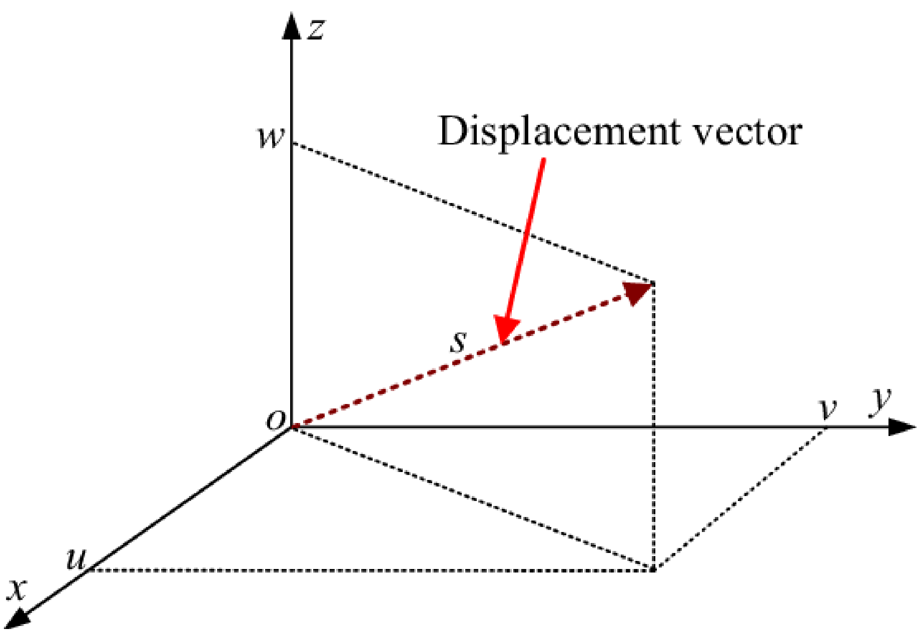
Determine the position vector of the point with coordinates (4, -1, 8).

$$\boxed{4\underline{i} - \underline{j} + 8\underline{k}}$$

Displacement Vectors

What is a displacement vector?

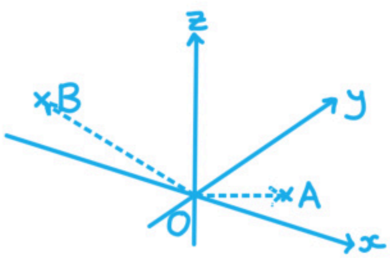
- A **displacement vector** describes the shortest route between any two points
  - It describes the **direction** and the **distance** between any two points
  - It is different to a **position vector** which describes the direction and distance from the point O: 0**i +** 0**j** or $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- The displacement vector of point B from the point A is written with the notation $\overrightarrow{AB}$
- A displacement vector between two points can be written in terms of the displacement vectors of a third point
  - $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$
- A displacement vector can be written in terms of its position vectors
  - For example the displacement vector $\overrightarrow{AB}$ can be written in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$
  - $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$
  - For position vector **a** = $\overrightarrow{OA}$ and **b** = $\overrightarrow{OB}$ the displacement vector $\overrightarrow{AB}$ can be written **b** – **a**



Worked example

The point A has coordinates (3, 0, -1) and the point B has coordinates (-2, -5, 7). Find the displacement vector $\overrightarrow{AB}$.

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix}$$



$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

$$\boxed{\overrightarrow{AB} = \begin{pmatrix} -5 \\ -5 \\ 8 \end{pmatrix}}$$

3.9.3 Magnitude of a Vector

Magnitude of a Vector

How do you find the magnitude of a vector?

- The **magnitude** of a vector tells us its **size** or **length**
  - For a **displacement** vector it tells us the **distance** between the two points
  - For a **position** vector it tells us the **distance** of the point from the **origin**
- The magnitude of the vector $\overrightarrow{\mathrm{AB}}$ is denoted $\left|\overrightarrow{\mathrm{AB}}\right|$
  - The magnitude of the vector **a** is denoted |**a**|
- The magnitude of a vector can be found using **Pythagoras' Theroem**
- The magnitude of a vector $\boldsymbol{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is found using
  - $|\boldsymbol{v}| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$
  - where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
  - This is **given in the formula booklet**

How do I find the distance between two points?

- **Vectors** can be used to find the distance (or displacement) between two points
  - It is the **magnitude** of the vector between them
- Given the **position vectors** of two points:
  - Find the displacement vector between them
  - Find the magnitude of the displacement vector between them

Worked example

Find the magnitude of the vector AB = 4**i** – **j** + 2**k**.

Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$\left|\overrightarrow{AB}\right| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$$

$$\boxed{\left|\overrightarrow{AB}\right| = \sqrt{21}}$$

Unit Vectors

What is a unit vector?

- A **unit vector** has a **magnitude** of 1
- It can be found by dividing a vector by its **magnitude**
  - This will result in a vector with a size of 1 unit in the direction of the original vector
- A unit vector in the direction of **a** is denoted $\frac{\mathbf{a}}{|\mathbf{a}|}$
  - For example a unit vector in the direction 3**i** – 4**j** is $\frac{(3\mathbf{i}-4\mathbf{j})}{\sqrt{3^2+4^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

Examiner Tip

- Finding the unit vector will not be a question on its own but will be a useful skill for further vectors problems so it is important to be confident with it

Worked example

Find the unit vector in the direction 2**i** – 2**j** + **k**.

Let $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$

Find the magnitude of $\underline{a}$

Magnitude of a vector	$\lvert v \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\lvert \underline{a} \rvert = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

Divide $\underline{a}$ by its magnitude:

Unit vector $= \frac{\underline{a}}{\lvert \underline{a} \rvert} = \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$

$\boxed{\frac{2}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{1}{3}\underline{k}}$

3.9.4 The Scalar Product

The Scalar ('Dot') Product

What is the scalar product?

- The scalar product (also known as the dot product) is one form in which two vectors can be combined together
- The scalar product between two vectors **a** and **b** is denoted $\mathbf{a} \cdot \mathbf{b}$
- The result of taking the scalar product of two vectors is a **real number**
  - i.e. a scalar
- The scalar product of two vectors gives information about the angle between the two vectors
  - If the scalar product is **positive** then the angle between the two vectors is **acute** (less than 90°)
  - If the scalar product is **negative** then the angle between the two vectors is **obtuse** (between 90° and 180°)
  - If the scalar product is **zero** then the angle between the two vectors is **90°** (the two vectors are **perpendicular**)

How is the scalar product calculated?

- There are **two methods** for calculating the scalar product
- The most common method used to find the scalar product between the two vectors ***v*** and ***w*** is to find the **sum of the product of each component** in the two vectors
  - $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The scalar product is also equal to the **product of the magnitudes** of the two vectors and the **cosine of the angle between them**
  - $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w| \cos \theta$
  - Where θ is the angle between ***v*** and ***w***
    - The two vectors ***v*** and ***w*** are joined at the start and pointing away from each other
- The scalar product can be used in the second formula to find the angle between the two vectors

What properties of the scalar product do I need to know?

- The order of the vectors doesn't change the result of the scalar product (it is **commutative**)
  - $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$
- The **distributive law** over addition can be used to 'expand brackets'
  - $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
- The scalar product is **associative** with respect to multiplication by a scalar

- $(k\boldsymbol{v}) \cdot (\boldsymbol{w}) = k(\boldsymbol{v} \cdot \boldsymbol{w})$
- The scalar product between a vector and itself is equal to the **square of its magnitude**
  - $\boldsymbol{v} \cdot \boldsymbol{v} = |\boldsymbol{v}|^2$
- If two vectors, ***v*** and ***w***, are **parallel** then the magnitude of the scalar product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \cdot \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because cos 0° = 1 and cos 180° = -1
- If two vectors are **perpendicular** the scalar product is **zero**
  - This is because cos 90° = 0

Worked example

Calculate the scalar product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{j} - 2\mathbf{k} - \mathbf{i}$ using:

i) the formula $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$,

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$

$\underline{w} = 3\underline{j} - 2\underline{k} - \underline{i} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

Be aware of the order of the terms.

Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = (2 \times -1) + (0 \times 3) + (-5 \times -2) = -2 + 10$

$\boxed{\underline{v} \cdot \underline{w} = 8}$

ii) the formula $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$, given that the angle between the two vectors is 66.6°.

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$ $\quad \underline{w} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$ $\quad |\underline{w}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$

$v \cdot w = \sqrt{29} \times \sqrt{14} \cos 66.6°$

$\boxed{\underline{v} \cdot \underline{w} = 8}$

Angle Between Two Vectors

How do I find the angle between two vectors?

- If two vectors with different directions are placed at the same starting position, they will form an angle between them
- The two formulae for the scalar product can be used together to find this angle
  - $\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}$
  - This is given in the formula booklet
- To find the angle between two vectors:
  - Calculate the scalar product between them
  - Calculate the magnitude of each vector
  - Use the formula to find $\cos\theta$
  - Use inverse trig to find θ

Worked example

Calculate the angle formed by the two vectors $\boldsymbol{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$\underline{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\underline{w} = 3\underline{i} + 4\underline{j} - \underline{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Start by finding the scalar product:

$\underline{v} \cdot \underline{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

$= (-1 \times 3) + (3 \times 4) + (2 \times -1) = 7$

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

$|\underline{w}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$

Angle between two vectors	$\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$

$\cos\theta = \dfrac{7}{\sqrt{14} \times \sqrt{26}} = 0.3668\ldots$

$\theta = \cos^{-1}(0.3668\ldots)$

$\boxed{\theta = 68.5^\circ \text{ (3sf)}}$

Perpendicular Vectors

How do I know if two vectors are perpendicular?

- If the **scalar product** of two (non-zero) vectors is **zero** then they are **perpendicular**
  - If $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then ***v*** and ***w*** must be perpendicular to each other
- Two vectors are **perpendicular** if their **scalar product** is **zero**
  - The value of cos θ = 0 therefore |***v***||***w***|cos θ = 0

Worked example

Find the value of *t* such that the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = (t-1)\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular to each other.

The two vectors $\underline{v}$ and $\underline{w}$ are perpendicular if $\underline{v} \cdot \underline{w} = 0$.

$$\underline{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} t-1 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{v} \cdot \underline{w} = 2(t-1) + t(-1) + 5(1)$$

$$= 2t - 2 - t + 5$$

Therefore $\underline{v}$ and $\underline{w}$ are perpendicular if

$$t + 3 = 0$$

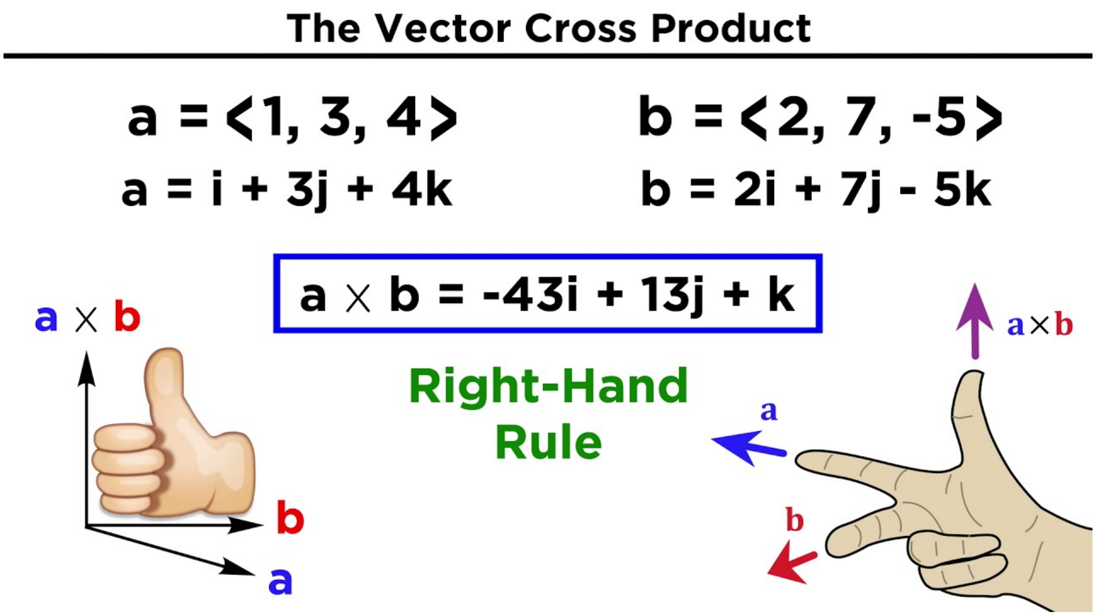
$$\boxed{t = -3}$$

3.9.5 The Vector Product

The Vector ('Cross') Product

What is the vector (cross) product?

- The **vector product** (also known as the **cross** product) is a form in which two vectors can be combined together
- The vector product between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted $\boldsymbol{v} \times \boldsymbol{w}$
- The result of taking the vector product of two vectors is a **vector**
- The **vector product** is a vector **in a plane** that is **perpendicular** to the two vectors from which it was calculated
  - This could be in either direction, depending on the angle between the two vectors
  - The **right-hand** rule helps you see which direction the vector product goes in
    - By pointing your index finger and your middle finger in the direction of the two vectors your thumb will automatically go in the direction of the vector product



How do I find the vector (cross) product?

- There are **two methods** for calculating the vector product
- The **vector product** of the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ can be written in **component form** as follows:

- $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$
- Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The vector product can also be found in terms of its **magnitude** and **direction**
- The **magnitude of the vector product** is equal to the **product of the magnitudes** of the two vectors and the **sine of the angle between them**
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |v||w|\sin\theta$
  - Where θ is the angle between ***v*** and ***w***
    - The two vectors ***v*** and ***w*** are joined at the start and pointing away from each other
  - This is **given in the formula booklet**
- The **direction of the vector product** is **perpendicular** to both ***v*** and ***w***

What properties of the vector product do I need to know?

- The order of the vectors **is important** and **changes the result** of the vector product
  - $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$
  - However
  - $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$
- The **distributive law** can be used to ‘expand brackets’
  - $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
  - Where ***u***, ***v*** and ***w*** are all vectors
- Multiplying a **scalar** by a vector gives the result:
  - $(k\boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{v} \times (k\boldsymbol{w}) = k(\boldsymbol{v} \times \boldsymbol{w})$
- The vector product between a vector and itself is equal to **zero**
  - $\boldsymbol{v} \times \boldsymbol{v} = 0$
- If two vectors are **parallel** then the vector product is **zero**
  - This is because sin 0° = sin 180° = 0
- If $\boldsymbol{v} \times \boldsymbol{w} = 0$ then ***v*** and ***w*** are parallel if they are non-zero
- If two vectors, ***v*** and ***w***, are **perpendicular** then the magnitude of the vector product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because sin 90° = 1

Worked example

Calculate the magnitude of the vector product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and

$\boldsymbol{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ using

i) the formula $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$,

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Use the formula to find the cross-product:

$$\underline{v} \times \underline{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} (0)(-1) - (-5)(-2) \\ (-5)(3) - (2)(-1) \\ (2)(-2) - (0)(3) \end{pmatrix} = \begin{pmatrix} -10 \\ -13 \\ -4 \end{pmatrix}$$

Find the magnitude of $\underline{v} \times \underline{w}$:

$$|\underline{v} \times \underline{w}| = \sqrt{(-10)^2 + (-13)^2 + (-4)^2} = \sqrt{285}$$

$$\boxed{|\underline{v} \times \underline{w}| = 16.9 \text{ (3sf)}}$$

ii) the formula , given that the angle between them is 1 radian.

Find the magnitude of $\underline{v}$ and $\underline{w}$:

$$|\underline{v}| = \sqrt{2^2 + 0^2 + (-5)^2} = \sqrt{29}$$

$$|\underline{w}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$$

$$|\underline{v} \times \underline{w}| = |\underline{v}||\underline{w}| \sin\theta$$

$$= \sqrt{29} \times \sqrt{14} \sin(1^c)$$

$$\boxed{|\underline{v} \times \underline{w}| = 17.0 \text{ (3sf)}}$$

Areas using Vector Product

How do I use the vector product to find the area of a parallelogram?

- The **area of the parallelogram** with two adjacent sides formed by the vectors ***v*** and ***w*** is equal to **the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = |\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **adjacent sides** of the parallelogram
    - This is **given in the formula booklet**

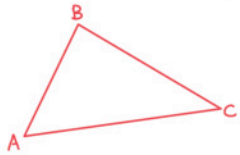
How do I use the vector product to find the area of a triangle?

- The **area of the triangle** with two sides formed by the vectors ***v*** and ***w*** is equal to **half of the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = \frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **sides** of the triangle
    - This is **not** given in the formula booklet

Worked example

Find the area of the triangle enclosed by the coordinates (1, 0, 5), (3, -1, 2) and (2, 0, -1).

Let A be (1,0,5), B be (3,-1,2) and C be (2,0,-1)



You can use any two direction vectors moving away from any vertex.

Find the two direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix}$$

Find the cross product of the two direction vectors:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} (-1)(-6) - (-3)(0) \\ (-3)(1) - (2)(-6) \\ (2)(0) - (-1)(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix}$$

Find the magnitude of the cross product

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{6^2 + 9^2 + 1^2} = \sqrt{118}$$

Area of the triangle is half the magnitude

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{118}$$

$$\boxed{\text{Area} = 5.43 \text{ u}^2 \text{ (3sf)}}$$

3.9.6 Geometric Proof with Vectors

Geometric Proof with Vectors

How can vectors be used to prove geometrical properties?

- If two vectors can be shown to be **parallel** then this can be used to prove parallel lines
  - If two vectors are **scalar multiples** of each other then they are **parallel**
  - To prove that two vectors are parallel simply show that one is a scalar multiple of the other
- If two vectors can be shown to be **perpendicular** then this can be used to prove perpendicular lines
  - If the **scalar product** is zero then the two vectors are **perpendicular**
- If two vectors can be shown to have equal **magnitude** then this can be used to prove two lines are the **same length**
- To prove a 2D shape is a **parallelogram** vectors can be used to
  - Show that there are two pairs of **parallel sides**
  - Show that the **opposite sides** are of **equal length**
    - The vectors opposite each other will be **equal**
  - If the angle between two of the vectors is shown to be 90° then the parallelogram is a **rectangle**
- To prove a 2D shape is a **rhombus** vectors can be used to
  - Show that there are two pairs of **parallel sides**
    - The vectors opposite each other with be **equal**
  - Show that **all four sides** are of **equal length**
  - If the angle between two of the vectors is shown to be 90° then the rhombus is a **square**

How are vectors used to follow paths through a diagram?

- In a geometric diagram the vector $\overrightarrow{AB}$ forms a path from the point A to the point B
  - This is specific to the path AB
  - If the vector $\overrightarrow{AB}$ is labelled **a** then any other vector with the same **magnitude** and **direction** as **a** could also be labelled **a**
- The vector $\overrightarrow{BA}$ would be labelled **–a**
  - It is **parallel** to **a** but pointing in the **opposite direction**
- If the point M is exactly halfway between A and B it is called the midpoint of A and the vector $\overrightarrow{AM}$ could be labelled $\frac{1}{2}\mathbf{a}$
- If there is a point X on the line AB such that $\overrightarrow{AX} = 2\overrightarrow{XB}$ then X is two-thirds of the way along the line $\overrightarrow{AB}$
  - Other ratios can be found in similar ways
  - A diagram often helps to visualise this
- If a point X divides a line segment AB into the ratio p : q then

- $\overrightarrow{AX} = \frac{p}{p+q}\overrightarrow{AB}$
- $\overrightarrow{XB} = \frac{q}{p+q}\overrightarrow{AB}$

How can vectors be used to find the midpoint of two vectors?

- If the point A has position vector **a** and the point B has position vector **b** then the **position vector** of the **midpoint** of $\overrightarrow{AB}$ is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$
  - The **displacement vector** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - Let **M** be the midpoint of $\overrightarrow{AB}$ then $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB}) = \frac{1}{2}(\mathbf{b}-\mathbf{a})$
  - The **position vector** $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b}-\mathbf{a}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$

How can vectors be used to prove that three points are collinear?

- Three points are collinear if they all **lie on the same line**
  - The vectors between the three points will be **scalar multiples** of each other
- The points A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$
- If the points A, B and M are collinear and $\overrightarrow{AM} = \overrightarrow{MB}$ then M is the **midpoint** of $\overrightarrow{AB}$

Worked example

Use vectors to prove that the points *A*, *B*, *C* and *D* with position vectors **a** = (3**i** – 5**j** – 4**k**), **b** = (8**i** - 7**j** - 5**k**), **c** = (3**i** - 2**j** + 4**k**) and **d** = (5**k** – 2**i**) are the vertices of a parallelogram.

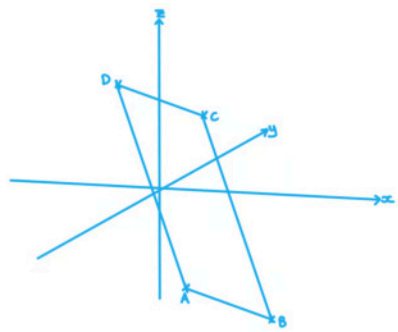
Find the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 9 \end{pmatrix}$$

$$\overrightarrow{CD} = \underline{d} - \underline{c} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{DA} = \underline{a} - \underline{d} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -9 \end{pmatrix}$$



$\overrightarrow{AB} = -\overrightarrow{CD}$ and $\overrightarrow{BC} = -\overrightarrow{DA}$ $\therefore$ ABCD must be a parallelogram