DP IB Maths: AI HL

3.9 Modelling with Vectors

Contents

3.9.1 Kinematics with Vectors

Kinematics using Vectors

How are vectors related to kinematics?

- Kinematics is the use of mathematics to model motion in objects
- If an object is moving in **one dimension** then its velocity, displacement and time are related using the formula $s = vt$
  - where s is **displacement**, v is **velocity** and t is the **time taken**
- If an object is moving in **more than one dimension** then **vectors** are needed to represent its **velocity** and **displacement**
  - Whilst **time** is a **scalar quantity**, **displacement** and **velocity** are both **vector quantities**
- Vectors are often used in questions in the context of **forces**, **acceleration** or **velocity**
- The position of an object at a particular time can be modelled using a vector equation

How do I find the direction of a vector?

- Vectors have opposite directions if they are the same size but opposite signs
- The direction of a vector is what makes it more than just a scalar
  - E.g. two objects with velocities of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- Two vectors are **parallel** if and only if one is a **scalar multiple** of the other
- For real-life contexts such as mechanics, direction can be calculated from a given vector using **trigonometry**
  - **Given the i and j components a right-triangle can be created and the angle found using SOHCAHTOA**
- It is usually given as a **bearing** or as an angle calculated **anticlockwise** from the positive x-axis

How do I find the distance between two moving objects?

- If two objects are moving with constant velocity in non-parallel directions the distance between them will change
- The distance between them can be found by finding the magnitude of their position vectors at any point in time
- The **shortest distance** between the two objects at a particular time can be found by finding the value of the time at which the magnitude is at its minimum value
  - Let the time when the objects are at the shortest distance be t
  - Find the distance, d, in terms of t by substituting into the equation for the magnitude of their position vectors
  - d^2 will be an expression in terms of t which can be differentiated and set to 0
  - Solving this will give the time at which the distance is at a minimum
  - Substitute this back into the expression for d to find the shortest distance

Worked example

Two objects, A and B, are moving so that their position relative to a fixed point, O at time t, in minutes can be defined by the position vectors $\boldsymbol{r}_A = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\boldsymbol{r}_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

The unit vectors **i** and **j** are a displacement of 1 metre due East and North of O respectively.

a) Find the coordinates of the initial position of the two objects.

The initial position is when $t = 0$

$$r_A = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 0\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$r_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 0\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\boxed{A\ (3, -1) \text{ and } B\ (2, 5)}$$

b) Find the shortest distance between the two objects and the time at which this will occur.

Let the shortest distance occur at time, t, then:

A: $(3-2t, -1+4t)$ B: $(2+3t, 5-t)$

Find the distance between A and B in terms of t:

$$d = \sqrt{[(2+3t)-(3-2t)]^2 + [(5-t)-(-1+4t)]^2}$$

$$= \sqrt{(-1+5t)^2 + (6-5t)^2}$$

$$= \sqrt{(1-10t+25t^2)+(36-60t+25t^2)}$$

$$d^2 = 37 - 70t + 50t^2$$

Find the minimum point of d^2:

$$\frac{dd^2}{dt} = -70 + 100t \quad \therefore \; -70+100t = 0$$

$$t = \frac{70}{100} = 0.7$$

When $t = 0.7$, $d = \sqrt{37 - 70(0.7) + 50(0.7)^2} = \sqrt{12.5}$

$$\boxed{d = 3.54 \text{ m (3 s.f.)}}$$

3.9.2 Constant & Variable Velocity

Vectors & Constant Velocity

How are vectors used to model linear motion?

- If an object is moving with **constant velocity** it will travel in a **straight line**
- For an object moving in a **straight line** in two or three dimensions its velocity, displacement and time can be related using the vector equation of a line
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
  - Letting
    - r be the position of the object at the time, t
    - $\boldsymbol{a}$ be the position vector, $\boldsymbol{r_0}$ at the start ($t = 0$)
    - λ represent the time, t
    - $\boldsymbol{b}$ be the **velocity** vector, $\boldsymbol{v}$
  - Then the position of the object at the time, t can be given by
    - $\boldsymbol{r} = \boldsymbol{r_0} + t\mathbf{v}$
- The **velocity vector** is the direction vector in the equation of the line
- The speed of the object will be the magnitude of the velocity $|\boldsymbol{v}|$

Worked example

A car, moving at constant speed, takes 2 minutes to drive in a straight line from point A (-4, 3) to point B (6, -5).

At time t, in minutes, the position vector ($\boldsymbol{p}$) of the car relative to the origin can be given in the form $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{b}$.

Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Vector $\underline{a}$ represents the initial position and vector $\underline{b}$ represents the direction vector per minute.

Position vector $\overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

At $t = 0$ minutes, $\underline{p} = \underline{a}$ so $\underline{a} = \overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Position vector $\overrightarrow{OB} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

At $t = 2$ minutes, the car is at the point B and so $\overrightarrow{OB} = \underline{a} + 2\underline{b}$

$\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + 2\underline{b}$

Direction vector $2\underline{b} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$

$\boxed{\underline{a} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}}$

Vectors & Variable Velocity

How are vectors used to model motion with variable velocity?

- The **velocity** of a particle is the rate of change of its **displacement** over time
- In one dimension velocity, ***v***, is found be taking the derivative of the displacement, ***s***, with respect to time, ***t***
  - $v = \frac{\mathrm{d}s}{\mathrm{d}t}$
- In more than one dimension **vectors** are used to represent motion
- For displacement given as a function of time in the form
  - $\mathbf{r}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$
- The velocity vector can be found by differentiating each component of the vector individually
  - $\mathbf{v} = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$
  - $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} f_1{}'(t) \\ f_2{}'(t) \end{pmatrix}$
  - The velocity should be left as a **vector**
  - The speed is the magnitude of the velocity
- If the velocity vector is known, displacement can be found by **integrating** each component of the vector individually
  - The constant of integration for each component will need to be found
- The **acceleration** of a particle is the rate of change of its **velocity** over time
- In one dimension acceleration, ***a***, is found be taking the derivative of the velocity, ***v***, with respect to time, ***t***
  - $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$
- In two dimensions acceleration can be found by differentiating each component of the velocity vector individually
  - $\mathbf{a} = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$
  - $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \begin{pmatrix} v_1{}'(t) \\ v_2{}'(t) \end{pmatrix}$
  - $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} f_1{}''(t) \\ f_2{}''(t) \end{pmatrix}$

- If the acceleration vector is known, the velocity vector can be found by **integrating** each component of the acceleration vector individually
  - The constant of integration for each component will need to be found

Worked example

A ball is rolling down a hill with velocity $v = \begin{pmatrix}5\\3\end{pmatrix} + t\begin{pmatrix}0\\-0.8\end{pmatrix}$. At the time $t = 0$ the position vector of the ball is $3\boldsymbol{i}-2\boldsymbol{j}$.

a) Find the acceleration vector of the ball's motion.

$$\underline{v} = \begin{pmatrix}5\\3-0.8t\end{pmatrix} \Rightarrow \underline{a} = \frac{d\underline{v}}{dt} = \begin{pmatrix}0\\-0.8\end{pmatrix}$$

$$\boxed{\underline{a} = -0.8\underline{j}}$$

b) Find the position vector of the ball at the time, t.

$$\underline{r} = \int \underline{v}\,dt = \int \begin{pmatrix}5\\3-0.8t\end{pmatrix} dt = \begin{pmatrix}5t + c\\3t - \frac{0.8t^2}{2} + d\end{pmatrix}$$

A constant of integration is needed for both components.

at $t = 0$, $\underline{r} = \begin{pmatrix}3\\-2\end{pmatrix}$

$$\begin{pmatrix}5(0) + c\\3(0) - \frac{0.8(0)^2}{2} + d\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix} \quad \therefore \quad c = 3,\ d = -2$$

$$\boxed{\underline{r} = (5t+3)\underline{i} + (3t - 0.4t^2 - 2)\underline{j}}$$