DP IB Maths: AA HL

3.8 Further Trigonometry

Contents

3.8.1 Trigonometric Proof

Trigonometric Proof

How do I prove new trigonometric identities?

- You can use trigonometric identities you already know to prove new identities
- Make sure you know how to find all of the trig identities in the formula booklet
  - The identity for tan, simple Pythagorean identity and the double angle identities for in and cos are in the SL section
    - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - $\cos^2\theta + \sin^2\theta = 1$
    - $\sin 2\theta = 2\sin\theta\cos\theta$
    - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
  - The reciprocal trigonometric identities for sec and cosec, further Pythagorean identities, compound angle identities and the double angle formula for tan
    - $\sec\theta = \dfrac{1}{\cos\theta}$
    - $\text{cosec }\theta = \dfrac{1}{\sin\theta}$
    - $1 + \tan^2\theta = \sec^2\theta$
    - $1 + \cot^2\theta = \text{cosec}^2\theta$
    - $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$
    - $\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$
    - $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
    - $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$
  - The identity for cot is **not in the formula booklet**, you will need to remember it
    - $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$
- To prove an identity start on one side and proceed step by step until you get to the other side
  - It is more common to start on the left hand side but you can start a proof from either end
  - Occasionally it is easier to show that one side subtracted from the other is zero
  - You should not work on both sides simultaneously

What should I look out for when proving new trigonometric identities?

- Look for anything that could be a part of one of the above identities on either side
  - For example if you see $\sin 2\theta$ you can replace it with $2\sin\theta\cos\theta$
  - If you see $2\sin\theta\cos\theta$ you can replace it with $\sin 2\theta$
- Look for ways of reducing the number of different trigonometric functions there are within the identity
  - For example if the identity contains tan θ, cot θ and cosec θ you could try
    - using the identities tan θ = 1/cot θ and 1 + $\cot^2\theta = \text{cosec}^2\theta$ to write it all in terms of cot θ
    - or rewriting it all in terms of sin θ and cos θ and simplifying
- Often you may need to trial a few different methods before finding the correct one
- Clever substitution into the **compound angle formulae** can be a useful tool for proving identities
  - For example rewriting $\cos\frac{\theta}{2}$ as $\cos\left(\theta - \frac{\theta}{2}\right)$ doesn't change the ratio but could make an identity easier to prove
- You will most likely need to be able to work with fractions and fractions-within-fractions
- Always keep an eye on the 'target' expression – this can help suggest what identities to use

Worked example

Prove that $8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$.

It is easiest to start on the right-hand side and apply the double angle formula for $\cos 2\theta$.

$$8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$$

The form of the left-hand side suggests that the identity $\cos 2A = 2\cos^2 A - 1$ would be more useful than the other options.

$$\cos 4\theta = 2\cos^2 2\theta - 1$$
$$= 2(2\cos^2\theta - 1)^2 - 1$$
$$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$$
$$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1$$
$$\therefore \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

3.8.2 Strategy for Trigonometric Equations

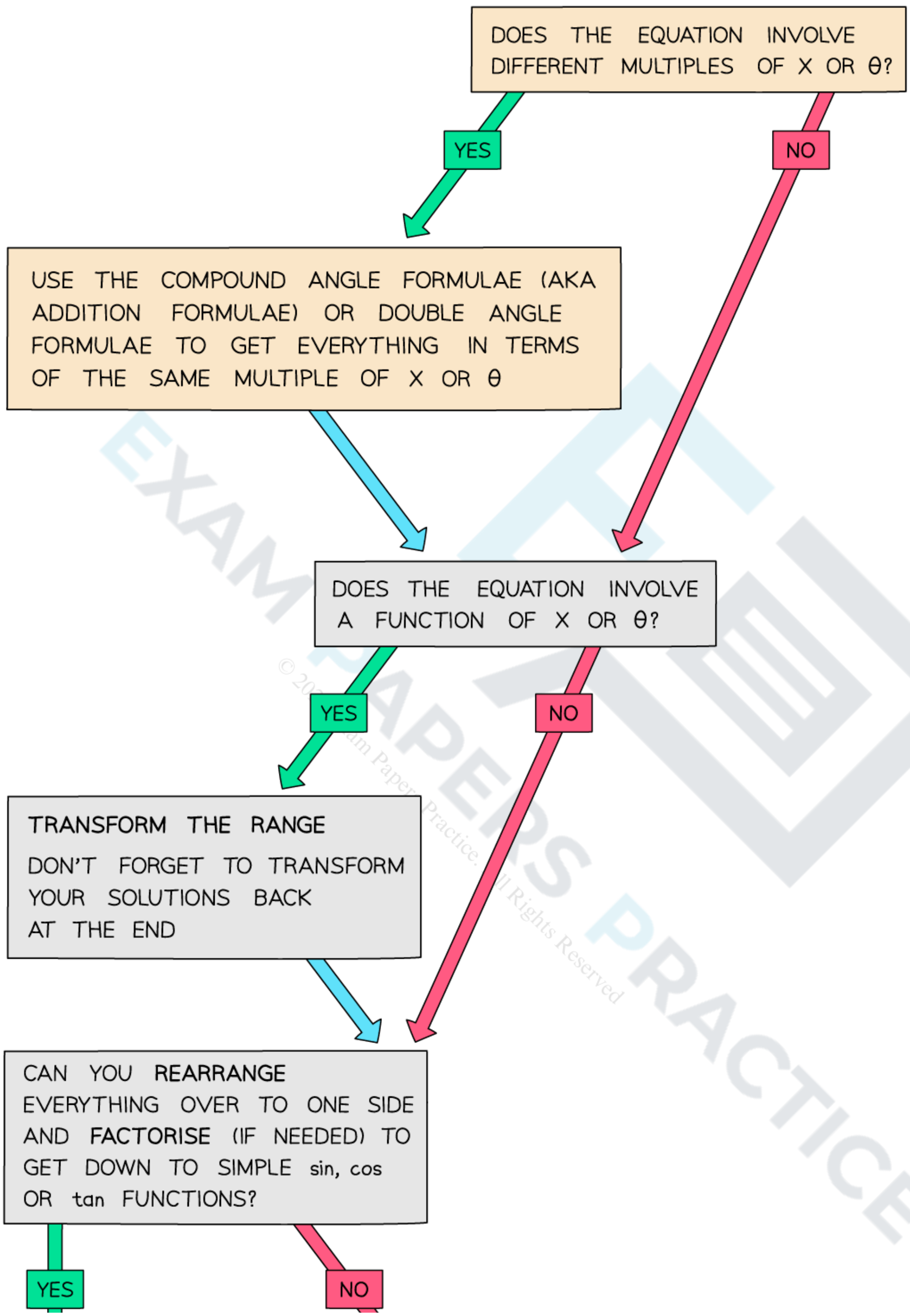
Strategy for Trigonometric Equations

How do I approach solving trig equations?

- You can solve trig equations in a variety of different ways
  - **Sketching a graph**
    - If you have your GDC it is always worth sketching the graph and using this to analyse its features
  - Using **trigonometric** identities**, Pythagorean** identities**,** the **compound** or **double angle** identities
    - Almost all of these are in the formula booklet, make sure you have it open at the right page
  - Using the **unit circle**
  - Factorising **quadratic** trig equations
    - Look out for quadratics such as $5\tan^2 x - 3\tan x - 4 = 0$
- The final rearranged equation you solve will involve **sin**, **cos** or **tan**
  - Don't try to solve an equation with **cosec**, **sec**, or **cot** directly

What should I look for when solving trig equations?

- Check the value of x or θ
  - If it is just x or θ you can begin solving
  - If there are **different multiples** of x or θ you will need to use the **double angle formulae** to get everything in terms of the same multiple of x or θ
  - If it is a **function** of x or θ, e.g. $2x - 15$, you will need to **transform the range** first
    - You must remember to transform your solutions back again at the end
- Does it involve more than one trigonometric function?
  - If it does, try to **rearrange** everything to bring it to one side, you may need to **factorise**
  - If not, can you use an identity to reduce the number of different trigonometric functions?
    - You should be able to use identities to reduce everything to just one simple trig function (either sin, cos or tan)
- Is it **linear** or **quadratic**?
  - If it is linear you should be able to rearrange and solve it
  - If it is quadratic you may need to factorise first
    - You will most likely get two solutions, consider whether they both **exist**
    - Remember solutions to $\sin x = k$ and $\cos x = k$ only exist for $-1 \le k \le 1$ whereas solutions to $\tan x = k$ exist for all values of k
- Are my solutions within the given range and do I need to find more solutions?
  - Be extra careful if your solutions are negative but the given range is positive only
  - Use a sketch of the graph or the unit circle to find the other solutions within the range
  - If you have a function of x or θ make sure you are finding the solutions within the **transformed range**
    - Don't forget to transform the solutions back so that they are in the required range at the end


DOES THE EQUATION INVOLVE DIFFERENT MULTIPLES OF X OR θ?
YES
NO
USE THE COMPOUND ANGLE FORMULAE (AKA ADDITION FORMULAE) OR DOUBLE ANGLE FORMULAE TO GET EVERYTHING IN TERMS OF THE SAME MULTIPLE OF X OR θ
DOES THE EQUATION INVOLVE A FUNCTION OF X OR θ?
YES
NO
TRANSFORM THE RANGE
DON'T FORGET TO TRANSFORM YOUR SOLUTIONS BACK AT THE END
CAN YOU REARRANGE EVERYTHING OVER TO ONE SIDE AND FACTORISE (IF NEEDED) TO GET DOWN TO SIMPLE sin, cos OR tan FUNCTIONS?
YES
NO

DOES IT INVOLVE AN IDENTITY?

$\tan x \equiv \frac{\sin x}{\cos x}$ OR $\sin^2 x + \cos^2 x \equiv 1$

USE THESE TO CONVERT EVERYTHING INTO ONE TRIG FUNCTION

FURTHER IDENTITIES

$1 + \tan^2 x \equiv \sec^2 x$

$1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

$\cot x \equiv \frac{1}{\tan x} \equiv \frac{\cos x}{\sin x}$

IS IT **LINEAR** OR **QUADRATIC**?

LINEAR

CAN I REARRANGE IT? IS THERE A COMMON FACTOR?

QUADRATIC

CAN I REARRANGE IT? DO I NEED TO FACTORISE? YOU SHOULD END UP WITH TWO SOLUTIONS

IF YOU HAVE SOLVED YOUR EQUATION FOR sec, cosec OR cot, USE THE IDENTITIES

$\sec x \equiv \frac{1}{\cos x}$ $\operatorname{cosec} x \equiv \frac{1}{\sin x}$ $\cot x \equiv \frac{1}{\tan x}$

TO CONVERT TO cos, sin OR tan BEFORE TRYING TO SOLVE FOR X OR θ

DO ALL MY SOLUTIONS ACTUALLY EXIST?

REMEMBER SOLUTIONS TO $\sin x = k$ AND $\cos x = k$ ONLY EXIST WHERE $-1 \leqslant k \leqslant 1$ WHEREAS SOLUTIONS TO $\tan x = k$ EXIST FOR **ALL VALUES** OF k

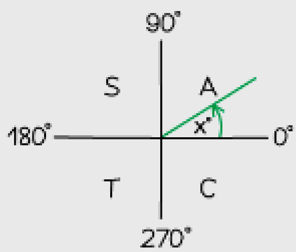
FIND THE **PRINCIPLE VALUE**

ARE SOLUTIONS WITHIN GIVEN RANGE?
DO I HAVE ALL SOLUTIONS?

YES

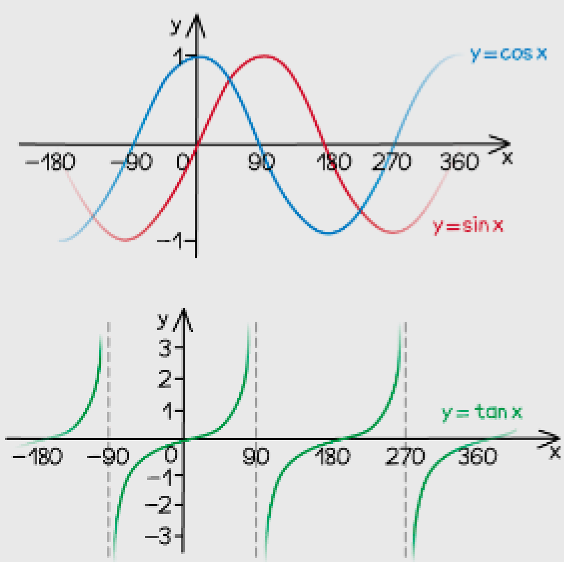
NO

USE CAST



SKETCH THE GRAPH

USE YOUR KNOWLEDGE OF GRAPH SYMMETRIES AND PERIODICITY TO FIND ALL OTHER SOLUTIONS



CHECK WHETHER YOU HAVE FOUND **ALL SOLUTIONS** AND THAT THEY ARE **WITHIN THE GIVEN RANGE**
(DON'T FORGET TO TRANSFORM YOUR SOLUTIONS BACK AT THE END IF THEY WERE FROM A TRANSFORMED RANGE)

Worked example

Find the solutions of the equation $(1+\cot^2 2\theta)(5\cos^2\theta - 1) = \cot^2 2\theta$ in the interval $0 \le \theta \le 2\pi$.

Move equivalent trig functions to the same sides:

$$5\cos^2\theta - 1 = \frac{\cot^2 2\theta}{1+\cot^2 2\theta}$$

divide both sides by $1+\cot^2 2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$$5\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) - 1 = \frac{\cot^2 2\theta}{\mathrm{cosec}^2 2\theta}$$

$$\frac{5}{2}\cos 2\theta + \frac{3}{2} = \frac{\frac{\cos^2 2\theta}{\sin^2 2\theta}}{\frac{1}{\sin^2 2\theta}}$$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\mathrm{cosec}\theta = \frac{1}{\sin\theta}$

$$\frac{1}{2}(5\cos 2\theta + 3) = \cos^2 2\theta$$

Rearrange to form a quadratic in $\cos 2\theta$

$$2\cos^2 2\theta - 5\cos 2\theta - 3 = 0$$

$$(2\cos 2\theta + 1)(\cos 2\theta - 3) = 0$$

$\cos 2\theta = -\frac{1}{2}$ or $\cos 2\theta = 3$ ← no solutions

We are solving the equation for 2θ so we must transform the range first: $0 \le 2\theta \le 4\pi$

$$2\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \quad \text{(primary value)}$$

so $2\theta = \frac{2\pi}{3}$, $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$, $\frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$, $\frac{4\pi}{3} + 2\pi = \frac{10\pi}{3}$

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$