3.8 Vector Equations of Lines

IB Maths - Revision Notes

3.8.1 Vector Equations of Lines

Equation of a Line in Vector Form

How do I find the vector equation of a line?

- The formula for finding the **vector equation** of a line is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
    - Where ***r*** is the **position vector** of any point on the line
    - ***a*** is the **position vector** of a known point on the line
    - ***b*** is a **direction** (displacement) **vector**
    - λ is a scalar
  - This is **given in the formula booklet**
  - This equation can be used for vectors in both 2- and 3- dimensions
- This formula is similar to a regular equation of a straight line in the form $y = mx + c$ but with a vector to show both a point on the line and the direction (or gradient) of the line
  - In 2D the gradient can be found from the direction vector
  - In 3D a numerical value for the direction cannot be found, it is given as a vector
- As ***a*** could be the position vector of **any** point on the line and ***b*** could be **any scalar multiple** of the direction vector there are infinite vector equations for a single line
- Given any two points on a line with position vectors ***a*** and ***b*** the **displacement** vector can be written as ***b*** - ***a***
  - So the formula **r = a +** λ**(b - a)** can be used to find the vector equation of the line
  - This is **not given in the formula booklet**

How do I determine whether a point lies on a line?

- Given the equation of a line $\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ the point ***c*** with position vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ is on the line if there exists a value of λ such that
  - $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
  - This means that there exists a single value of λ that satisfies the three equations:
    - $c_1 = a_1 + \lambda b_1$

- $c_2 = a_2 + \lambda b_2$
- $c_3 = a_3 + \lambda b_3$

- A GDC can be used to solve this system of linear equations for
  - The point only lies on the line if a single value of λ exists for all three equations
- Solve one of the equations first to find a value of λ that satisfies the first equation and then check that this value also satisfies the other two equations
- If the value of λ does not satisfy all three equations, then the point $\boldsymbol{c}$ does not lie on the line

Exam Tip

- Remember that the vector equation of a line can take many different forms
  - This means that the answer you derive might look different from the answer in a mark scheme
- You can choose whether to write your vector equations of lines using unit vectors or as column vectors
  - Use the form that you prefer, however column vectors is generally easier to work with

Worked example

a) Find a vector equation of a straight line through the points with position vectors **a** = 4**i** – 5**k** and **b** = 3**i** - 3**k**

Use the position vectors to find the displacement vector between them.

$$\vec{OA} = \begin{pmatrix}4\\0\\-5\end{pmatrix}, \quad \vec{OB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} \Rightarrow \vec{AB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} - \begin{pmatrix}4\\0\\-5\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$

$$r = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix} \quad \text{or} \quad r = \begin{pmatrix}3\\0\\-3\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

(position vector of point a; direction vector) (position vector of point b; direction vector)

$$r = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

b) Determine whether the point C with coordinate (2, 0, -1) lies on this line.

Let $c = \begin{pmatrix}2\\0\\-1\end{pmatrix}$, then check to see if there exists a value of λ such that

$$\begin{pmatrix}2\\0\\-1\end{pmatrix} = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

From the 'i' component : $4 - \lambda = 2$ ①

From the 'j' component : $0 + 0\lambda = 0$ ② (✓) Works for all λ

From the 'k' component : $-5 + 2\lambda = -1$ ③

① $\Rightarrow \lambda = 2$ sub into ③ $\Rightarrow -5 + (2 \times 2) = -5 + 4 = -1$ ✓

Point C lies on the line

Equation of a Line in Parametric Form

How do I find the vector equation of a line in parametric form?

- By considering the three separate components of a vector in the *x*, *y* and *z* directions it is possible to write the **vector equation** of a line as **three separate equations**
  - Letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ becomes
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
    - Where $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a position vector and $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is a direction vector
  - This vector equation can then be split into its three separate component forms:
    - $x = x_0 + \lambda l$
    - $y = y_0 + \lambda m$
    - $z = z_0 + \lambda n$
  - These are **given in the formula booklet**

Worked example

Write the parametric form of the equation of the line which passes through the point (-2, 1, 0) with direction vector $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$.

Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

Use $r = a + \lambda b$ to write the equation in vector form first:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

position vector of a point — direction vector

Separate the components into their 3 separate equations.

$x = -2 + 3\lambda$
$y = 1 + \lambda$
$z = -4\lambda$

Angle Between Two Lines

How do we find the angle between two lines?

- The angle between two lines is equal to the angle between their **direction vectors**
  - It can be found using the **scalar product** of their direction vectors
- Given two lines in the form $\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_2$ use the formula
  - $\theta = \cos^{-1}\left(\dfrac{\boldsymbol{b}_1 \bullet \boldsymbol{b}_2}{|\boldsymbol{b}_1||\boldsymbol{b}_2|}\right)$
- If you are given the equations of the lines in a different form or two points on a line you will need to find their direction vectors first
- To find the angle ABC the vectors BA and BC would be used, both starting from the point B
- The intersection of two lines will always create **two angles**, an acute one and an obtuse one
  - A **positive scalar product** will result in the **acute angle** and a **negative scalar product** will result in the **obtuse angle**
    - Using the **absolute value** of the scalar product will **always result in the acute angle**

Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Worked example

Find the acute angle, in radians between the two lines defined by the equations:

$$l_1:\ \boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \text{ and } l_2:\ \boldsymbol{b} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$$

STEP 1: Find the scalar product of the direction vectors:

$$\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = (1 \times -3) + (-4 \times 2) + (-3 \times 5) = -3 + (-8) + (-15) = -26$$

negative, so the angle will be the obtuse angle.

STEP 2: Find the magnitudes of the direction vectors:

$$\sqrt{(1)^2 + (-4)^2 + (-3)^2} = \sqrt{26} \qquad \sqrt{(-3)^2 + (2)^2 + (5)^2} = \sqrt{38}$$

STEP 3: Find the angle:

$$\cos\theta = \frac{|-26|}{\sqrt{26}\sqrt{38}}$$

Using the absolute value will result in the acute angle

$$\theta = \cos^{-1}\left(\frac{26}{\sqrt{26}\sqrt{38}}\right)$$

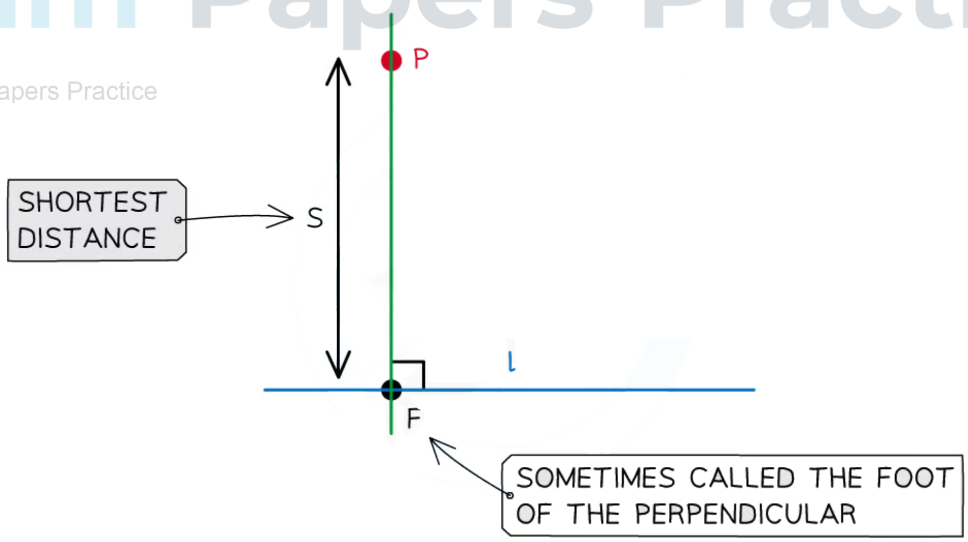
$\theta = 0.597$ radians (3sf)

3.8.2 Shortest Distances with Lines

Shortest Distance Between a Point and a Line

How do I find the shortest distance from a point to a line?

- The shortest distance from any point to a line will always be the **perpendicular** distance
  - Given a line *l* with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a point *P* not on *l*
  - The **scalar product** of the direction vector, **b**, and the vector in the direction of the **shortest distance** will be zero
- The shortest distance can be found using the following steps:
  - STEP 1: Let the vector equation of the line be *r* and the point not on the line be *P*, then the point on the line closest to *P* will be the point *F*
    - The point *F* is sometimes called the foot of the perpendicular
  - STEP 2: Sketch a diagram showing the line *l* and the points *P* and *F*
    - The vector $\overrightarrow{FP}$ will be **perpendicular** to the line *l*
  - STEP 3: Use the equation of the line to find the position vector of the point *F* in terms of λ
  - STEP 4: Use this to find the displacement vector $\overrightarrow{FP}$ in terms of λ
  - STEP 5: The scalar product of the direction vector of the line *l* and the displacement vector $\overrightarrow{FP}$ will be zero
    - Form an equation $\overrightarrow{FP} \cdot \mathbf{b} = 0$ and solve to find λ
  - STEP 6: Substitute λ into $\overrightarrow{FP}$ and find the magnitude $\left|\overrightarrow{FP}\right|$
    - The shortest distance from the point to the line will be the magnitude of $\overrightarrow{FP}$
- Note that the shortest distance between the point and the line is sometimes referred to as the **length of the perpendicular**



How do we use the vector product to find the shortest distance from a point to a line?

- The vector product can be used to find the shortest distance from any point to a line on a 2-dimensional plane
- Given a point, P, and a line $\mathbf{r} = a + \lambda b$
  - The shortest distance from P to the line will be $\dfrac{|\overrightarrow{AP} \times b|}{|b|}$
  - Where A is a point on the line
  - This is **not** given in the formula booklet

Exam Tip

- Column vectors can be easier and clearer to work with when dealing with scalar products.

Worked example

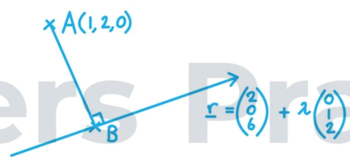
Point A has coordinates (1, 2, 0) and the line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Point B lies on the l such that $[AB]$ is perpendicular to l.

Find the shortest distance from A to the line l.

B is on l so can be written in terms of λ:

$$\overrightarrow{OB} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix}$$



Find $\overrightarrow{AB}$ using $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l: $\overrightarrow{AB} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\lambda - 2 + 2(6 + 2\lambda) = 0$$

$$5\lambda + 10 = 0$$

$$\lambda = -2$$

Substitute back into $\overrightarrow{AB}$ and find the magnitude:

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2-2 \\ 6+2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$$

Shortest distance $= \sqrt{21}$ units

Shortest Distance Between Two Lines

How do we find the shortest distance between two parallel lines?

- Two **parallel** lines will never intersect
- The shortest distance between two **parallel lines** will be the **perpendicular distance** between them
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and a line l_2 with equation $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ then the shortest distance between them can be found using the following steps:
  - STEP 1: Find the vector between $\mathbf{a}_1$ and a general coordinate from l_2 in terms of μ
  - STEP 2: Set the scalar product of the vector found in STEP 1 and the direction vector $\mathbf{d}_1$ equal to zero
    - Remember the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are scalar multiples of each other and so either can be used here
  - STEP 3: Form and solve an equation to find the value of μ
  - STEP 4: Substitute the value of μ back into the equation for l_2 to find the coordinate on l_2 closest to l_1
  - STEP 5: Find the distance between $\mathbf{a}_1$ and the coordinate found in STEP 4
- Alternatively, the formula $\frac{\left|\overrightarrow{AB} \times \mathbf{d}\right|}{|\mathbf{d}|}$ can be used
  - Where $\overrightarrow{AB}$ is the vector connecting the two given coordinates $\mathbf{a}_1$ and $\mathbf{a}_2$
  - **d** is the simplified vector in the direction of $\mathbf{d}_1$ and $\mathbf{d}_2$
  - This is **not** given in the formula booklet

How do we find the shortest distance from a given point on a line to another line?

- The shortest distance from any point on a line to another line will be the **perpendicular** distance from the point to the line
- If the angle between the two lines is known or can be found then right-angled trigonometry can be used to find the perpendicular distance
  - The formula $\frac{\left|\overrightarrow{AB} \times \mathbf{d}\right|}{|\mathbf{d}|}$ given above is derived using this method and can be used
- Alternatively, the equation of the line can be used to find a general coordinate and the steps above can be followed to find the shortest distance

How do we find the shortest distance between two skew lines?

- Two **skew** lines are not parallel but will never intersect
- The shortest distance between two **skew lines** will be perpendicular to **both** of the lines

- This will be at the point where the two lines pass each other with the perpendicular distance where the point of intersection would be
- The **vector product** of the two direction vectors can be used to find a vector in the direction of the shortest distance
- The shortest distance will be a vector **parallel** to the vector product

- To find the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{d}_2$,
  - STEP 1: Find the vector product of the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$
    - $\mathbf{d} = \mathbf{d}_1 \times \mathbf{d}_2$
  - STEP 2: Find the vector in the direction of the line between the two general points on l_1 and l_2 in terms of λ and μ
    - $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - STEP 3: Set the two vectors parallel to each other
    - $\mathbf{d} = k\overrightarrow{AB}$
  - STEP 4: Set up and solve a system of linear equations in the three unknowns, k, λ and μ

Exam Tip

- Exam questions will often ask for the shortest, or minimum, distance within vector questions
- If you're unsure start by sketching a quick diagram
- Sometimes calculus can be used, however vector methods are usually required

Worked example

A drone travels in a straight line and at a constant speed. It moves from an initial point (-5, 4, -8) in the direction of the vector $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. At the same time as the drone begins moving a bird takes off from initial point (6, -4, 3) and moves in a straight line at a constant speed in the direction of the vector $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

Find the minimum distance between the bird and the drone during this movement.

Find the vector product of the direction vectors.

$$\begin{pmatrix}2\\-3\\4\end{pmatrix}\times\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}(-3)(1)-(4)(2)\\(4)(-1)-(2)(1)\\(2)(2)-(-3)(-1)\end{pmatrix}=\begin{pmatrix}-11\\-6\\1\end{pmatrix}$$

Find the vector in the direction of the line between the general coordinates.

$$\overrightarrow{AB}=\begin{pmatrix}-5-\mu\\4+2\mu\\-8+\mu\end{pmatrix}-\begin{pmatrix}6+2\lambda\\-4-3\lambda\\3+4\lambda\end{pmatrix}=\begin{pmatrix}-11-\mu-2\lambda\\8+2\mu+3\lambda\\-11+\mu-4\lambda\end{pmatrix}$$

A point on l_2 A point on l_1

$$\begin{pmatrix}-11-\mu-2\lambda\\8+2\mu+3\lambda\\-11+\mu-4\lambda\end{pmatrix}=k\begin{pmatrix}-11\\-6\\1\end{pmatrix}$$

$\overrightarrow{AB}$ is parallel to $\begin{pmatrix}-11\\-6\\1\end{pmatrix}$ so $\overrightarrow{AB}=k\begin{pmatrix}-11\\-6\\1\end{pmatrix}$

Set up and solve a system of equations.

$$\begin{cases}11k-2\lambda-\mu=11\\6k+3\lambda+2\mu=-8\\\mu-4\lambda-k=11\end{cases}$$

Solve using GDC: $k=\frac{31}{79}$ $\lambda=-\frac{238}{79}$ $\mu=-\frac{52}{79}$

Substitute back into the expression for $\overrightarrow{AB}$ and find the magnitude:

$$|\overrightarrow{AB}|=\left|\begin{pmatrix}-11-\left(-\frac{52}{79}\right)-2\left(-\frac{238}{79}\right)\\8+2\left(-\frac{52}{79}\right)+3\left(-\frac{238}{79}\right)\\-11+\left(-\frac{52}{79}\right)-4\left(-\frac{238}{79}\right)\end{pmatrix}\right|=\left|\begin{pmatrix}-\frac{341}{79}\\-\frac{186}{79}\\\frac{31}{79}\end{pmatrix}\right|=\sqrt{\left(-\frac{341}{79}\right)^2+\left(-\frac{186}{79}\right)^2+\left(\frac{31}{79}\right)^2}$$

Shortest distance = 4.93 units (3 s.f.)