DP IB Maths: AA HL

3.7 Inverse & Reciprocal Trig Functions

Contents

3.7.1 Reciprocal Trig Functions

Reciprocal Trig Functions

What are the reciprocal trig functions?

- There are three reciprocal trig functions that each correspond to either sin, cos or tan
  - Secant (sec *x*)
    - $\sec x = \dfrac{1}{\cos x}$
  - Cosecant (cosec *x*)
    - $\text{cosec } x = \dfrac{1}{\sin x}$
  - Cotangent (cot *x*)
    - $\cot x = \dfrac{1}{\tan x}$
  - The identities above for sec *x* and cosec *x* are given in the formula booklet
  - The identity for cot *x* is **not given**, you will need to remember it
  - A good way to remember which function is which is to look at the **third** letter in each of the reciprocal trig functions
    - co**t** *x* is 1 over **t**an *x* etc
- Each of the reciprocal trig functions are undefined for certain values of *x*
  - sec *x* is undefined for values of *x* for which cos *x* = 0
  - cosec *x* is undefined for values of *x* for which sin *x* = 0
  - cot *x* is undefined for values of *x* for which tan *x* = 0
    - When tan *x* is undefined, cot *x* = 0
- Rearranging the identity $\tan x = \dfrac{\sin x}{\cos x}$ gives
  - $\cot x = \dfrac{\cos x}{\sin x}$
    - This is not in the formula booklet but is easily derived
- Be careful not to confuse the reciprocal trig functions with the inverse trig functions
  - $\sin^{-1} x \neq \dfrac{1}{\sin x}$

What do the graphs of the reciprocal trig functions look like?

- The graph of ***y* = sec*x*** has the following properties:
  - *The* ***y*-axis** is a **line of symmetry**
  - It has a **period** of **360°** (**2π** radians)
  - There are vertical **asymptotes** wherever **cos** *x* **= 0**
  - **If drawing the graph without the help of a GDC it is a good idea to sketch cos *x* first and draw these in**
- **The domain** is all *x* **except odd multiples of 90°** (90°, -90°, 270°, -270°, etc.)
  - in **radians** this is all *x* **except odd multiples of π/2** (π/2, - π/2, 3π/2, -3π/2, etc.)
- **The range** is ***y* ≤ -1 or y ≥ 1**

Worked example

Without the use of a calculator, find the values of

a) $\sec \dfrac{\pi}{6}$

$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)}$$

the third letter is c so sec is related to cos

$\cos\left(\frac{\pi}{6}\right)$ is an exact value you should know.

$$= \frac{1}{\frac{\sqrt{3}}{2}}$$

$$\boxed{\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}}$$

b) $\cot 45°$

$$\cot 45° = \frac{1}{\tan 45°}$$

the third letter is t so cot is related to tan

$\tan 45°$ is an exact value you should know.

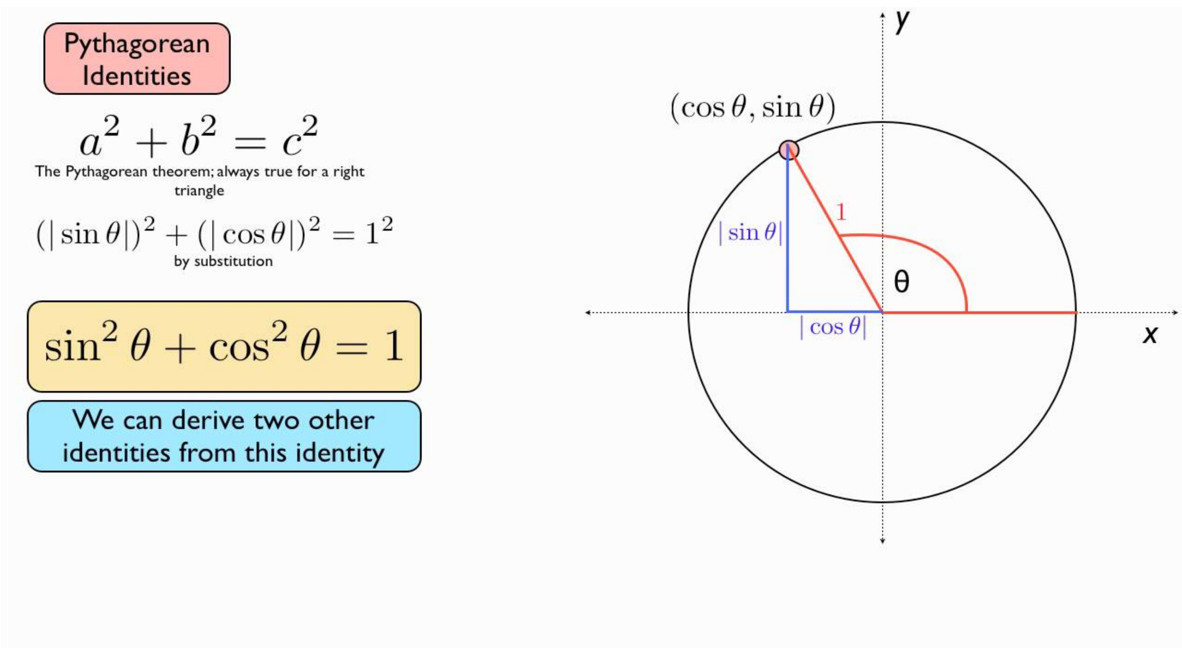
$$= \frac{1}{1}$$

$$\boxed{\cot 45° = 1}$$

Pythagorean Identities

What are the Pythagorean Identities?

- Aside from the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ there are two further Pythagorean identities you will need to learn
  - $1 + \tan^2\theta = \sec^2\theta$
  - $1 + \cot^2\theta = \text{cosec}^2\theta$
  - Both can be found in the formula booklet
- Both of these identities can be derived from $\sin^2 x + \cos^2 x = 1$
  - To derive the identity for **$\sec^2 x$** divide $\sin^2 x + \cos^2 x = 1$ by **$\cos^2 x$**
  - To derive the identity for **$\text{cosec}^2 x$** divide $\sin^2 x + \cos^2 x = 1$ by **$\sin^2 x$**



Worked example

Solve the equation $9\sec^2\theta - 11 = 3\tan\theta$ in the interval $0 \le \theta \le 2\pi$.

$$9\sec^2\theta - 11 = 3\tan\theta, \quad 0 \le \theta \le 2\pi$$

consider how this could be changed to use $\tan^2 + 1 = \sec^2$

Range is given in terms of π so work in radians

$$(9\sec^2\theta - 9) - 2 = 3\tan\theta$$

$$9(\sec^2\theta - 1) - 2 = 3\tan\theta$$

$$9\tan^2\theta - 3\tan\theta - 2 = 0$$

$$(3\tan\theta - 2)(3\tan\theta + 1) = 0$$

$\tan\theta = \frac{2}{3} \Rightarrow \theta = 0.5880\ldots$

or $\theta = \pi + 0.5880\ldots = 3.729\ldots$

or $\tan\theta = -\frac{1}{3} \Rightarrow \theta = -0.3217\ldots$

or $\theta = \pi + (-0.3217\ldots) = 2.819\ldots$

and $\theta = 2\pi + (-0.3217\ldots) = 5.961\ldots$

$$\theta = 0.588, 2.82, 3.73, 5.96 \ (3 \text{ s.f.})$$

3.7.2 Inverse Trig Functions

Inverse Trig Functions

What are the inverse trig functions?

- The functions **arcsin**, **arccos** and **arctan** are the **inverse functions** of **sin**, **cos** and **tan** respectively when their domains are restricted
  - $\sin(\arcsin x) = x$ for $-1 \le x \le 1$
  - $\cos(\arccos x) = x$ for $-1 \le x \le 1$
  - $\tan(\arctan x) = x$ for all x
- You will have seen and used the inverse trig **operations** many times already
  - Arcsin is the operation $\mathbf{sin^{-1}}$
  - Arccos is the operation $\mathbf{cos^{-1}}$
  - Arctan is the operation $\mathbf{tan^{-1}}$
- The domains of **sin**, **cos**, and **tan** must first be restricted to make them **one-to-one functions**
  - A function can only have an inverse if it is a one-to-one function
- The domain of **sin *x*** is restricted to $\mathbf{-\pi/2 \le x \le \pi/2}$ $\mathbf{(-90° \le x \le 90°)}$
- The domain of **cos *x*** is restricted to $\mathbf{0 \le x \le \pi}$ $\mathbf{(0° \le x \le 180°)}$
- The domain of **tan *x*** is restricted to $\mathbf{-\pi/2 < x < \pi/2}$ $\mathbf{(-90° < x < 90°)}$
- Be aware that $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ are **not** the same as the reciprocal trig functions
  - They are used to solve trig equations such as $\sin x = 0.5$ for all values of x
  - $\arcsin x$ is the same as $\sin^{-1} x$ but not the same as $(\sin x)^{-1}$

What do the graphs of the inverse trig functions look like?

- The graphs of **arcsin**, **arccos** and **arctan** are the **reflections** of the graphs of **sin**, **cos** and **tan** (after their domains have been restricted) in the line $y = x$
  - The **domains** of $\arcsin x$ and $\arccos x$ are both $-1 \le x \le 1$
  - The **range** of $\arcsin x$ is $-\pi/2 \le y \le \pi/2$

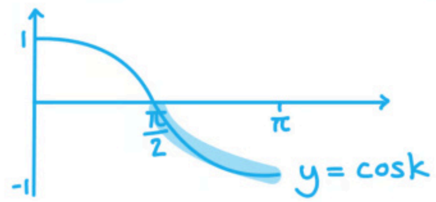
How are the inverse trig functions used?

- The functions **arcsin**, **arccos** and **arctan** are used to evaluate trigonometric equations such as $\sin x = 0.5$
  - If $\sin x = 0.5$ then $\arcsin 0.5 = x$ for values of x between $-\pi/2 \le x \le \pi/2$
    - You can then use symmetries of the trig function to find solutions over other intervals
- The inverse trig functions are also used to help evaluate algebraic expressions
  - From $\sin(\arcsin x) = x$ we can also say that $\sin^n(\arcsin x) = x^n$ for $-1 \le x \le 1$
  - If using an inverse trig function to evaluate an algebraic expression then remember to consider the domain and range of the function
    - $\arcsin(\sin x) = x$ only for $-\pi/2 \le x \le \pi/2$
    - $\arccos(\cos x) = x$ only for $0 \le x \le \pi$
    - $\arctan(\tan x) = x$ only for $-\pi/2 < x < \pi/2$
  - The symmetries of the trig functions can be used when values lie outside of the domain or range
    - Using $\sin(x) = \sin(\pi - x)$ you get $\arcsin(\sin(2\pi/3)) = \arcsin(\sin(\pi/3)) = \pi/3$

Worked example

Given that x satisfies the equation $\arccos x = k$ where $\frac{\pi}{2} < k < \pi$, state the range of possible values of x.

If $\arccos x = k$, then $x = \cos k$ $(\cos(\arccos x) = x)$



For $\frac{\pi}{2} < k < \pi$, $-1 < \cos k < 0$

$-1 < x < 0$