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3.7 Inverse & Reciprocal Trig Functions

IB Maths - Revision Notes

AA HL

3.7.1 Reciprocal Trig Functions

Reciprocal Trig Functions

What are the reciprocal trig functions?

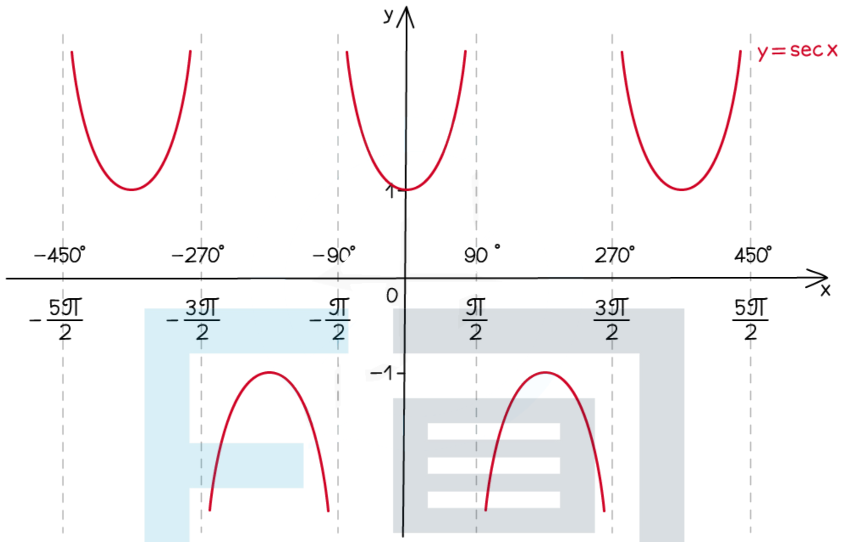
- There are three reciprocal trig functions that each correspond to either sin, cos or tan
  - Secant (sec *x*)
    - $\sec x = \dfrac{1}{\cos x}$
  - Cosecant (cosec *x*)
    - $\operatorname{cosec} x = \dfrac{1}{\sin x}$
  - Cotangent (cot *x*)
    - $\cot x = \dfrac{1}{\tan x}$
  - The identities above for sec *x* and cosec *x* are given in the formula booklet
  - The identity for cot *x* is **not given**, you will need to remember it
  - A good way to remember which function is which is to look at the **third** letter in each of the reciprocal trig functions
    - co**t** *x* is 1 over **t**an *x* etc
- Each of the reciprocal trig functions are undefined for certain values of *x*
  - sec *x* is undefined for values of *x* for which cos *x* = 0
  - cosec *x* is undefined for values of *x* for which sin *x* = 0
  - cot *x* is undefined for values of *x* for which tan *x* = 0
    - When tan *x* is undefined, cot *x* = 0

- Rearranging the identity $\tan x = \dfrac{\sin x}{\cos x}$ gives
  - $\cot x = \dfrac{\cos x}{\sin x}$
    - This is not in the formula booklet but is easily derived
- Be careful not to confuse the reciprocal trig functions with the inverse trig functions
  - $\sin^{-1} x \neq \dfrac{1}{\sin x}$

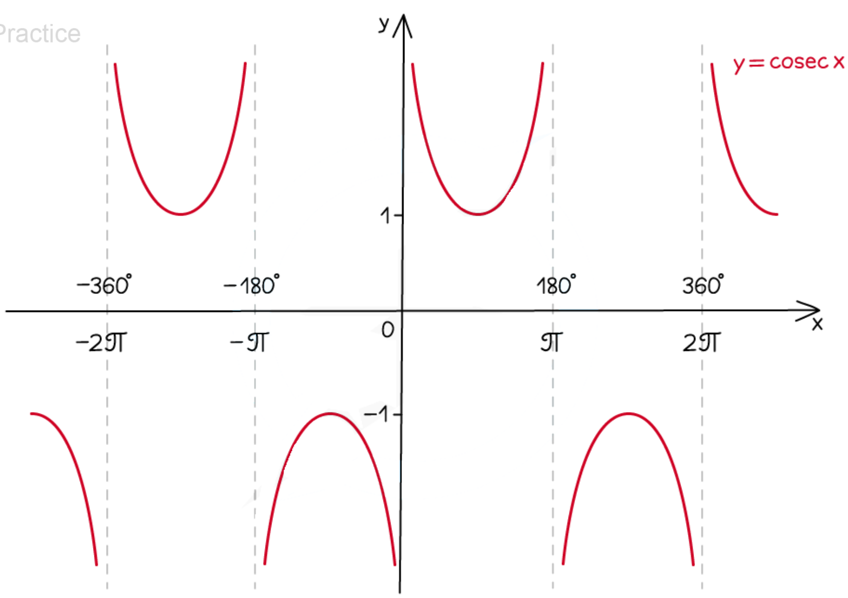
What do the graphs of the reciprocal trig functions look like?

- The graph of ***y* = sec *x*** has the following properties:

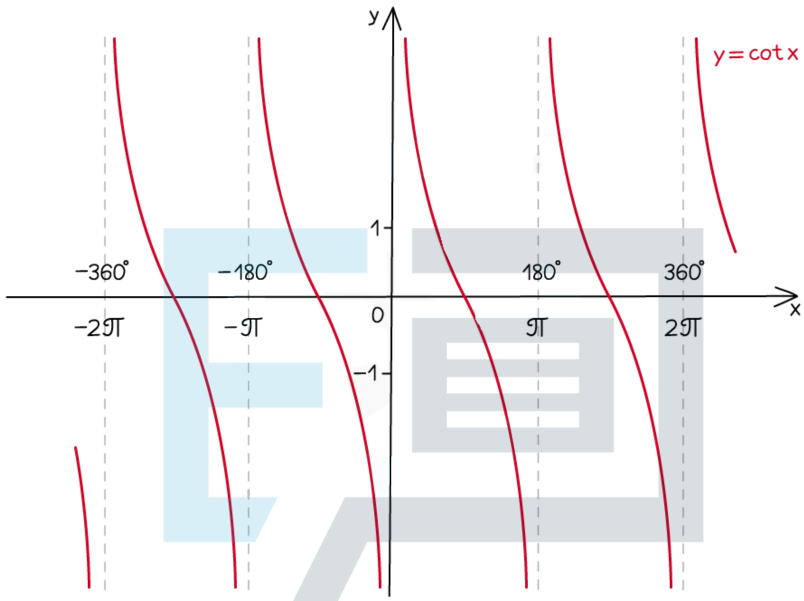
- *The* ***y*-axis** is a **line of symmetry**
- It has a **period** of **360° (2π** radians)
- There are vertical **asymptotes** wherever **cos** $x = 0$
  - **If drawing the graph without the help of a GDC it is a good idea to sketch cos *x* first and draw these in**
- **The domain** is all x **except odd multiples of 90°** (90°, -90°, 270°, -270°, etc.)
  - in **radians** this is all x **except odd multiples of π/2** (π/2, - π/2, 3π/2, -3π/2, etc.)
- **The range** is $y \le -1$ **or** $y \ge 1$



- The graph of ***y* = cosec *x*** has the following properties:
  - It has a **period** of **360° (2π** radians)
  - There are vertical **asymptotes** wherever **sin** $x = 0$
    - **If drawing the graph it is a good idea to sketch sin *x* first and draw these in**
  - The **domain** is all x **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
    - in radians this is all x **except multiples of π** (0, π, - π, 2π, -2π, etc.)
  - The range is $y \le -1$ or $y \ge 1$



- The graph of $y = \cot x$ has the following properties
  - It has a **period** of **180°** or **π** radians
  - There are vertical **asymptotes** wherever $\tan x = 0$
  - The **domain** is all x **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
    - In radians this is all x **except multiples of π** ($0, \pi, -\pi, 2\pi, -2\pi$, etc.)
  - The **range** is $y \in \mathbb{R}$ (i.e. cot can take *any* real number value)



Exam Tip

- To solve equations with the reciprocal trig functions, convert them into the regular trig functions and solve in the usual way
- Don't forget that both **tan** and **cot** can be written in terms of **sin** and **cos**
- You will sometimes see **csc** instead of **cosec** for cosecant

Worked example

Without the use of a calculator, find the values of

a) $\sec \dfrac{\pi}{6}$

$\sec\left(\frac{\pi}{6}\right) = \dfrac{1}{\cos\left(\frac{\pi}{6}\right)}$

the third letter is c so sec is related to cos

$\cos\left(\frac{\pi}{6}\right)$ is an exact value you should know.

$= \dfrac{1}{\frac{\sqrt{3}}{2}}$

$\boxed{\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}}$

b) $\cot 45°$

$\cot 45° = \dfrac{1}{\tan 45°}$

the third letter is t so cot is related to tan

$\tan 45°$ is an exact value you should know.

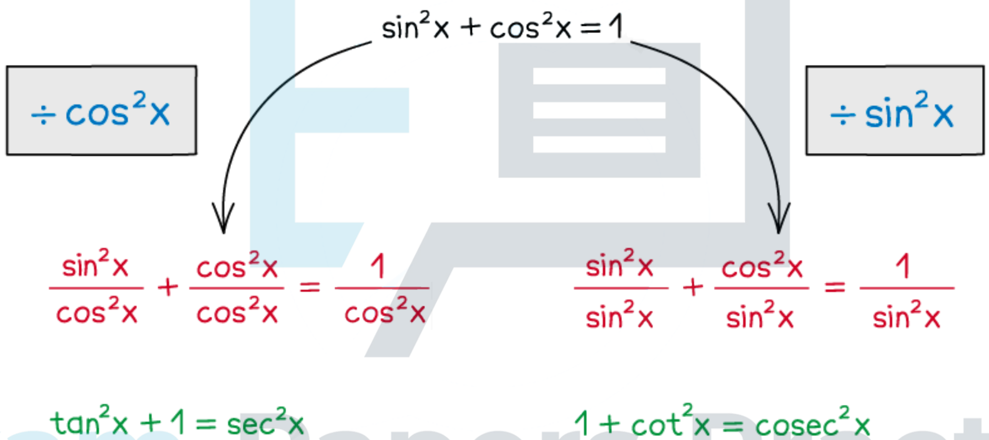
$= \dfrac{1}{1}$

$\boxed{\cot 45° = 1}$

Pythagorean Identities

What are the Pythagorean Identities?

- Aside from the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ there are two further Pythagorean identities you will need to learn
  - $1 + \tan^2\theta = \sec^2\theta$
  - $1 + \cot^2\theta = \mathrm{cosec}^2\theta$
  - Both can be found in the formula booklet
- Both of these identities can be derived from $\sin^2 x + \cos^2 x = 1$
  - To derive the identity for $\mathbf{sec^2} \boldsymbol{x}$ divide $\sin^2 x + \cos^2 x = 1$ by $\mathbf{cos^2} \boldsymbol{x}$
  - To derive the identity for $\mathbf{cosec^2} \boldsymbol{x}$ divide $\sin^2 x + \cos^2 x = 1$ by $\mathbf{sin^2} \boldsymbol{x}$



Exam Tip

All the Pythagorean identities can be found in the **Topic 3: Geometry and Trigonometry** section of the formula booklet

Worked example

Solve the equation $9\sec^2\theta - 11 = 3\tan\theta$ in the interval $0 \leq \theta \leq 2\pi$.

$9\sec^2\theta - 11 = 3\tan\theta, \quad 0 \leq \theta \leq 2\pi$

consider how this could be changed to use $\tan^2 + 1 = \sec^2$

Range is given in terms of π so work in radians

$(9\sec^2\theta - 9) - 2 = 3\tan\theta$

$9(\sec^2\theta - 1) - 2 = 3\tan\theta$

$9\tan^2\theta - 3\tan\theta - 2 = 0$

$(3\tan\theta - 2)(3\tan\theta + 1) = 0$

$\tan\theta = \frac{2}{3} \Rightarrow \theta = 0.5880...$

or $\theta = \pi + 0.5880... = 3.729...$

or $\tan\theta = -\frac{1}{3} \Rightarrow \theta = -0.3217...$

or $\theta = \pi + (-0.3217...) = 2.819...$

and $\theta = 2\pi + (-0.3217...) = 5.961...$

$\boxed{\theta = 0.588, 2.82, 3.73, 5.96 \text{ (3 s.f.)}}$

3.7.2 Inverse Trig Functions

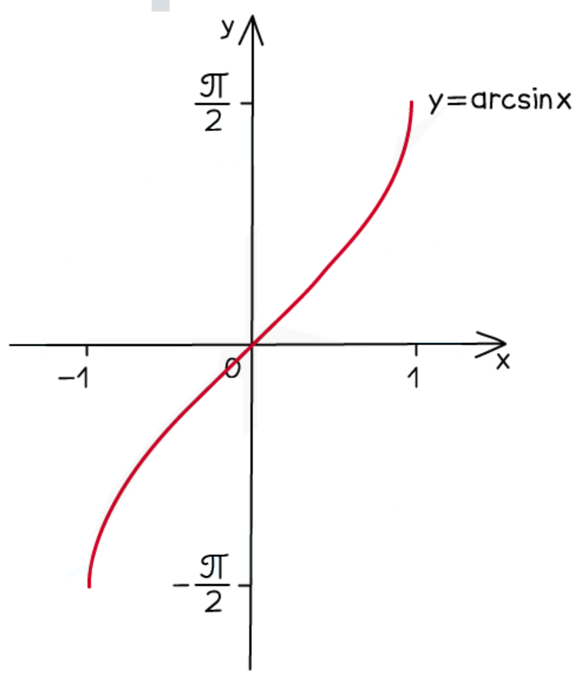
Inverse Trig Functions

What are the inverse trig functions?

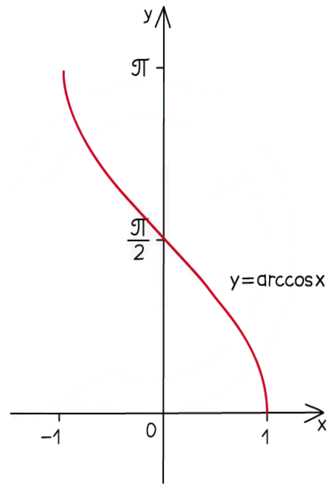
- The functions **arcsin**, **arccos** and **arctan** are the **inverse functions** of **sin**, **cos** and **tan** respectively when their domains are restricted
  - sin (arcsin x) = x for $-1 \le x \le 1$
  - cos (arccos x) = x for $-1 \le x \le 1$
  - tan (arctan x) = x for all x
- You will have seen and used the inverse trig **operations** many times already
  - Arcsin is the operation $\mathbf{sin^{-1}}$
  - Arccos is to the operation $\mathbf{cos^{-1}}$
  - Arctan is the operation $\mathbf{tan^{-1}}$
- The domains of **sin**, **cos**, and **tan** must first be restricted to make them **one-to-one functions**
  - A function can only have an inverse if it is a one-to-one function
- The domain of **sin *x*** is restricted to $-\pi/2 \le x \le \pi/2$ ($-90° \le x \le 90°$)
- The domain of **cos *x*** is restricted to $0 \le x \le \pi$ ($0° \le x \le 180°$)
- The domain of **tan *x*** is restricted to $-\pi/2 < x < \pi/2$ ($-90° < x < 90°$)
- Be aware that $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ are **not** the same as the reciprocal trig functions
  - They are used to solve trig equations such as sin x = 0.5 for all values of x
  - arcsin x is the same as $\sin^{-1} x$ but not the same as $(\sin x)^{-1}$

What do the graphs of the inverse trig functions look like?

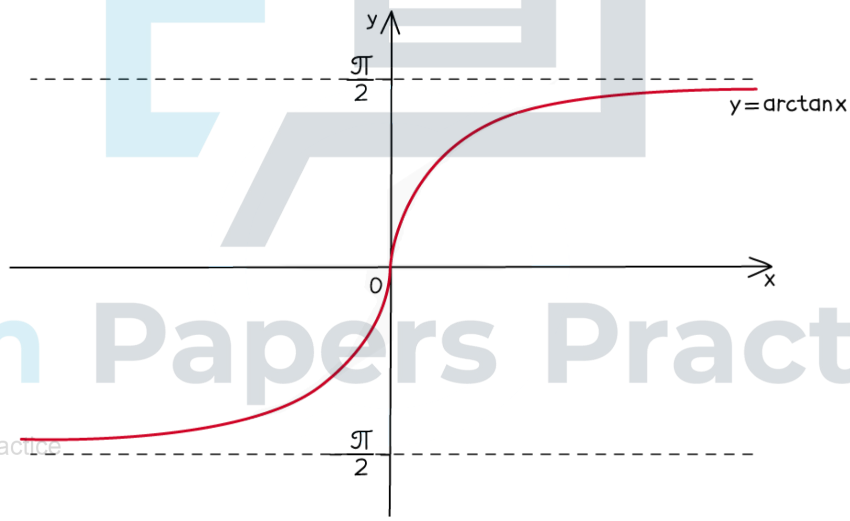
- The graphs of **arcsin**, **arccos** and **arctan** are the **reflections** of the graphs of **sin**, **cos** and **tan** (after their domains have been restricted) in the line $y = x$
  - The **domains** of arcsin x and arccos x are both $-1 \le x \le 1$
  - The **range** of arcsin x is $-\pi/2 \le y \le \pi/2$



- The **range** of arccos x is $0 \le y \le \pi$



- The **domain** of arctan x is $x \in \mathbb{R}$
- The **range** of arctan x is $-\pi/2 < y < \pi/2$
  - Note that there are horizontal asymptotes at $\pi/2$ and $-\pi/2$



How are the inverse trig functions used?

- The functions **arcsin**, **arccos** and **arctan** are used to evaluate trigonometric equations such as sin $x = 0.5$
  - If sin $x = 0.5$ then arcsin $0.5 = x$ for values of x between $-\pi/2 \le x \le \pi/2$
    - You can then use symmetries of the trig function to find solutions over other intervals
- The inverse trig functions are also used to help evaluate algebraic expressions
  - From sin (arcsin x) = x we can also say that $\sin^n(\arcsin x) = x^n$ for $-1 \le x \le 1$
  - If using an inverse trig function to evaluate an algebraic expression then remember to consider the domain and range of the function
    - $\arcsin(\sin x) = x$ only for $-\pi/2 \le x \le \pi/2$
    - $\arccos(\cos x) = x$ only for $0 \le x \le \pi$
    - $\arctan(\tan x) = x$ only for $-\pi/2 < x < \pi/2$

- The symmetries of the trig functions can be used when values lie outside of the domain or range
  - Using sin(x) = sin(π - x) you get arcsin(sin(2π/3)) = arcsin(sin(π/3)) = π/3

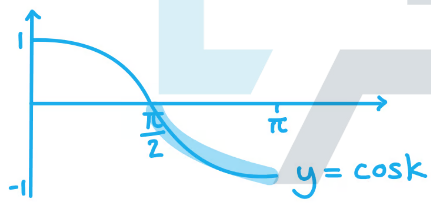
Exam Tip

- Make sure you know the shapes of the graphs for **sin**, **cos** and **tan** so that you can easily reflect them in the line $y = x$ and hence sketch the graphs of **arcsin**, **arccos** and **arctan**

Worked example

Given that x satisfies the equation $\arccos x = k$ where $\frac{\pi}{2} < k < \pi$, state the range of possible values of x.

If $\arccos x = k$, then $x = \cos k$ $(\cos(\arccos x) = x)$



For $\frac{\pi}{2} < k < \pi$, $-1 < \cos k < 0$

$\boxed{-1 < x < 0}$