Exam Papers Practice

3.6 Trigonometric Equations & Identities

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Course	DP IB Maths
Section	3. Geometry & Trigonometry
Topic	3.6 Trigonometric Equations & Identities
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1

(i) $\cos^2\alpha + \left(\frac{3}{7}\right)^2 = 1$ Use Pythagorean identity

$\cos^2\alpha = 1 - \left(\frac{3}{7}\right)^2 = \frac{40}{49} \implies \cos\alpha = \pm\sqrt{\frac{40}{49}} = \pm\frac{2\sqrt{10}}{7}$

$\boxed{\cos\alpha = \frac{2\sqrt{10}}{7}}$ } $\cos\alpha \geq 0$ for $0 \leq \alpha \leq \pi/2$

(ii) $\sin 2\alpha = 2\left(\frac{3}{7}\right)\left(\frac{2\sqrt{10}}{7}\right)$ Use double angle identity

$\boxed{\sin 2\alpha = \frac{12\sqrt{10}}{49}}$

(iii) $\cos 2\alpha = 1 - 2\left(\frac{3}{7}\right)^2$ Use double angle identity

$1 - 2\sin^2\alpha$ is easiest here!

$\boxed{\cos 2\alpha = \frac{31}{49}}$

(iv) $\tan 2\alpha = \dfrac{\left(\frac{12\sqrt{10}}{49}\right)}{\left(\frac{31}{49}\right)}$ Use identity for tangent

$\boxed{\tan 2\alpha = \frac{12\sqrt{10}}{31}}$

Question 2

$\left(\frac{1}{5}\right)^2 + \sin^2 B = 1$ Use Pythagorean identity to find $\sin B$

$\sin^2 B = 1 - \left(\frac{1}{5}\right)^2 = \frac{24}{25} \implies \sin B = \pm\sqrt{\frac{24}{25}} = \pm\frac{2\sqrt{6}}{5}$

$\implies \sin B = -\frac{2\sqrt{6}}{5}$ $\Big\}$ $\sin B \leq 0$ for $\frac{3\pi}{2} \leq B \leq 2\pi$

(i) $\cos 2B = 2\left(\frac{1}{5}\right)^2 - 1$ Use double angle identity

$\boxed{\cos 2B = -\frac{23}{25}}$

$2\cos^2 B - 1$ is easiest here!

(ii) $\sin 2B = 2\left(-\frac{2\sqrt{6}}{5}\right)\left(\frac{1}{5}\right)$ Use double angle identity

$\boxed{\sin 2B = -\frac{4\sqrt{6}}{25}}$

(iii) $\tan 2B = \dfrac{\left(-\frac{4\sqrt{6}}{25}\right)}{\left(-\frac{23}{25}\right)}$ Use identity for tangent

$\boxed{\tan 2B = \frac{4\sqrt{6}}{23}}$

Question 3

(i) $2\sin M\cos M = \sin 2M$ Use double angle identity

$2r\cos M = s$

$\boxed{\cos M = \frac{s}{2r}}$ This is valid as long as $r = \sin M \neq 0$

Note : $\cos^2 M + \sin^2 M = 1 \Rightarrow \cos^2 M + r^2 = 1$

$\Rightarrow \cos^2 M = 1 - r^2 \Rightarrow \cos M = \pm\sqrt{1-r^2}$

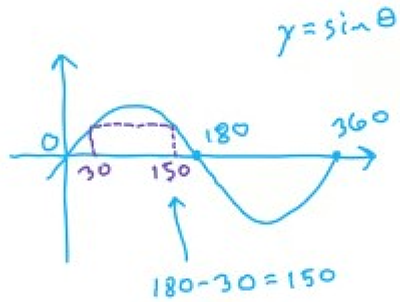
But we don't know if $\cos M$ is positive or negative, so we can't answer the question this way.

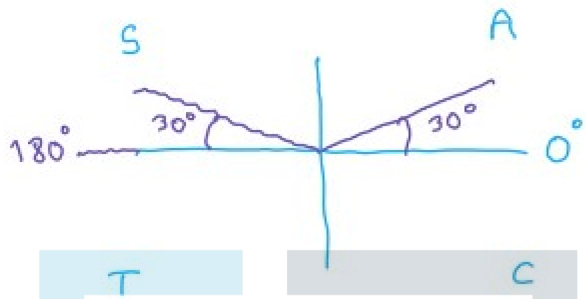
(ii) $\tan M = \frac{r}{\left(\frac{s}{2r}\right)}$ Use identity for tangent

$= r \div \frac{s}{2r} = r \times \frac{2r}{s}$

$\boxed{\tan M = \frac{2r^2}{s}}$

Question 4





$0 \le \theta \le 360 \implies 0 \le 2\theta \le 720$ start by transforming the interval

Find solutions for 2θ in the transformed interval:

$2\sin 2\theta = 1 \implies \sin 2\theta = 1/2$

$2\theta = \sin^{-1}(1/2) = 30°$ principal solution

$180 - 30 = 150$ use symmetry or CAST to find other solution between 0° and 360°

So $2\theta = 150°$ is also a solution

$30 + 360 = 390 \quad 150 + 360 = 510$ sine function repeats every 360°

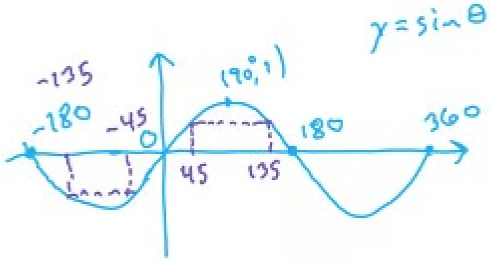
So $2\theta = 390°$ or $510°$ are also solutions solutions between 360° and 720°

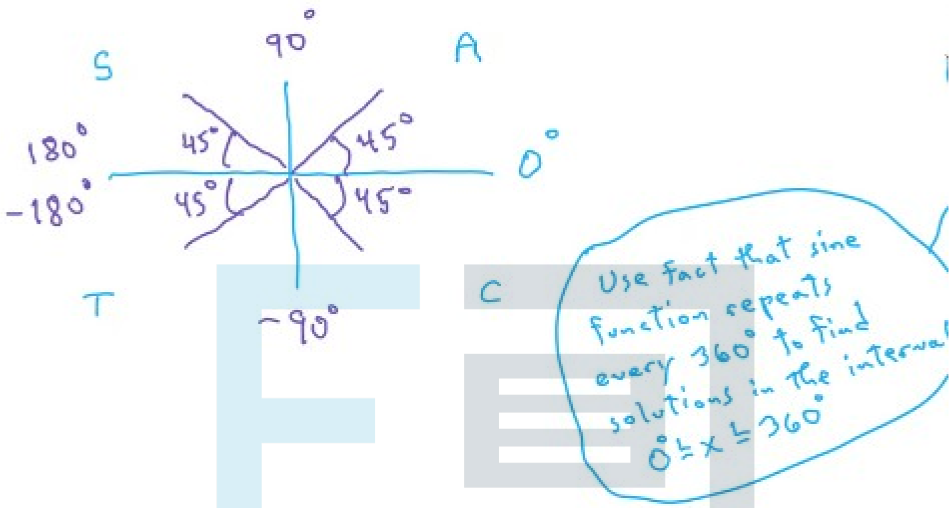
As FINAL step, convert to solutions for θ:

$\frac{30}{2} = 15 \quad \frac{150}{2} = 75 \quad \frac{390}{2} = 195 \quad \frac{510}{2} = 255$

$$\boxed{\text{The solutions are } \theta = 15°,\ 75°,\ 195°,\ 255°}$$

Question 5


y = sin θ
(90°, 1)
-135
-180
-45
O
45
135
180
360


90°
S
A
180°
-180°
0°
45°
45°
45°
45°
T
C
-90°
Use fact that sine function repeats every 360° to find solutions in the interval 0° ≤ x ≤ 360°

$$2\sin x = \frac{1}{\sin x}$$

$$2\sin^2 x = 1$$

$$\sin^2 x = \frac{1}{2} \implies \sin x = \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}}$$

If $\sin x = \frac{1}{\sqrt{2}}$

$x = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$ principal solution

$180 - 45 = 135$ use symmetry or CAST to find other solution

So $x = 135°$ is another solution

If $\sin x = -\frac{1}{\sqrt{2}}$

$x = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -45°$ principal solution

$-180 + 45 = -135$ use symmetry or CAST to find other solution

So $x = -135°$ is another solution

→ $-45 + 360 = 315$ $\quad -135 + 360 = 225$

So $x = 225°$ and $x = 315°$ are also solutions

The solutions are
$x = 45°, 135°, 225°, 315°$

Question 6

a) $(x-2)(x-3) = x^2 - 3x - 2x + 6 = x^2 - 5x + 6$

Therefore

$$(x+1)(x-2)(x-3) = (x+1)(x^2-5x+6)$$

$$= x(x^2-5x+6) + 1(x^2-5x+6)$$

$$= x^3 - 5x^2 + 6x + x^2 - 5x + 6$$

$$= x^3 - 4x^2 + x + 6$$

b) Use the result from part (a):

$$\tan^3 x - 4\tan^2 x + \tan x + 6 = 0$$

$$\Rightarrow (\tan x + 1)(\tan x - 2)(\tan x - 3) = 0$$

$$\Rightarrow \tan x = -1, \ \tan x = 2, \text{ or } \tan x = 3$$

$-45°$ isn't in the solution interval

$x = \tan^{-1}(-1) = -45°$ principal value (from GDC)

or $-45 + 180 = 135°$

or $135 + 180 = 315°$

find other solutions in interval

$x = \tan^{-1}(2) = 63.434948... = 63.4°$ (1 d.p.) principal value (from GDC)

or $63.4 + 180 = 243.4°$ (1 d.p.) find other solutions in interval

$x = \tan^{-1}(3) = 71.565051... = 71.6°$ (1 d.p.) principal value (from GDC)

or $71.6 + 180 = 251.6°$ (1 d.p.) find other solutions in interval

$$\boxed{x = 63.4°, 71.6°, 135°, 243.4°, 251.6°, 315° \text{ (to 1 d.p.)}}$$

Question 7

$\sin^2 x + \cos^2 x \equiv 1 \implies \sin^2 x \equiv 1 - \cos^2 x$

a) $2\sin^2 x + 3\cos x = 0$

$2(1-\cos^2 x) + 3\cos x = 0$ substitute for $\sin^2 x$

$2 - 2\cos^2 x + 3\cos x = 0$ expand brackets

$$\boxed{\begin{gathered}2\cos^2 x - 3\cos x - 2 = 0 \\ (a=2,\ b=-3,\ c=-2)\end{gathered}}$$ rearrange

Note that

$-2\cos^2 x + 3\cos x + 2 = 0$ $(a=-2,\ b=3,\ c=2)$

is also a valid answer. But the version in green will be more useful in part (b).

b) $2\sin^2 x + 3\cos x = 0$

$\longrightarrow 2\cos^2 x - 3\cos x - 2 = 0$ from part (a)

Let $y = \cos x$ rewrite quadratic in terms of y

$$\left.\begin{aligned} 2y^2 - 3y - 2 &= 0 \\ (2y+1)(y-2) &= 0 \\ y = -\tfrac{1}{2} \quad &\text{or} \quad y = 2 \end{aligned}\right\} \text{solve quadratic}$$

So $\cos x = -\frac{1}{2}$ or $\cos x = 2$ → no solution, $\cos x$ is never greater than one!

$x = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$ principal solution

or $x = -120°$ use symmetry or CAST to find other solution in range

$$\boxed{\text{The solutions are } x = 120° \text{ or } -120°}$$

Question 8

a) Use Pythagorean identity

$$\cos^2 x + \sin^2 x = 1 \implies \cos^2 x = 1 - \sin^2 x$$

Therefore

$$2\cos^2 x - \sin x = 1$$

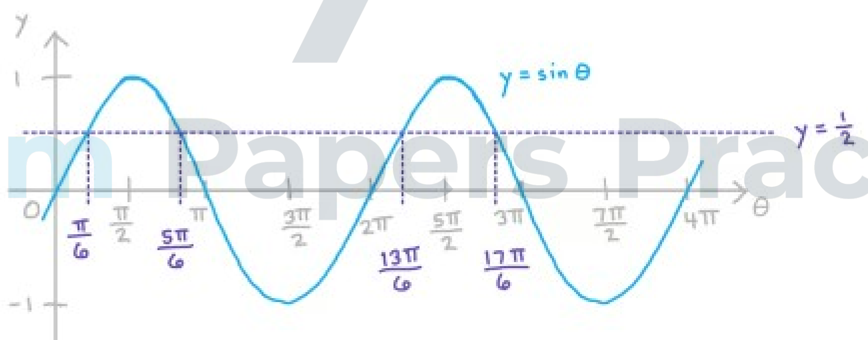
$$\implies 2(1 - \sin^2 x) - \sin x = 1$$

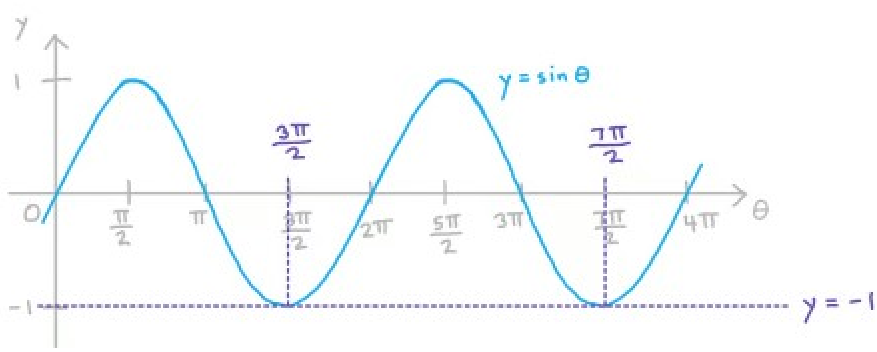
$$2 - 2\sin^2 x - \sin x = 1$$

$$0 = 1 - 2 + 2\sin^2 x + \sin x$$

$$0 = 2\sin^2 + \sin x - 1$$

$$\boxed{2\sin^2 x + \sin x - 1 = 0}$$





b) Use the result from part (a):

$$2\cos^2 x - \sin x = 1$$

$$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$$ This is a 'hidden quadratic'

$$(2\sin x - 1)(\sin x + 1) = 0$$ factorise

$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$$

Find primary solutions, then use symmetry of sine function to find other solutions in the interval:

$\sin x = \frac{1}{2}$: $x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

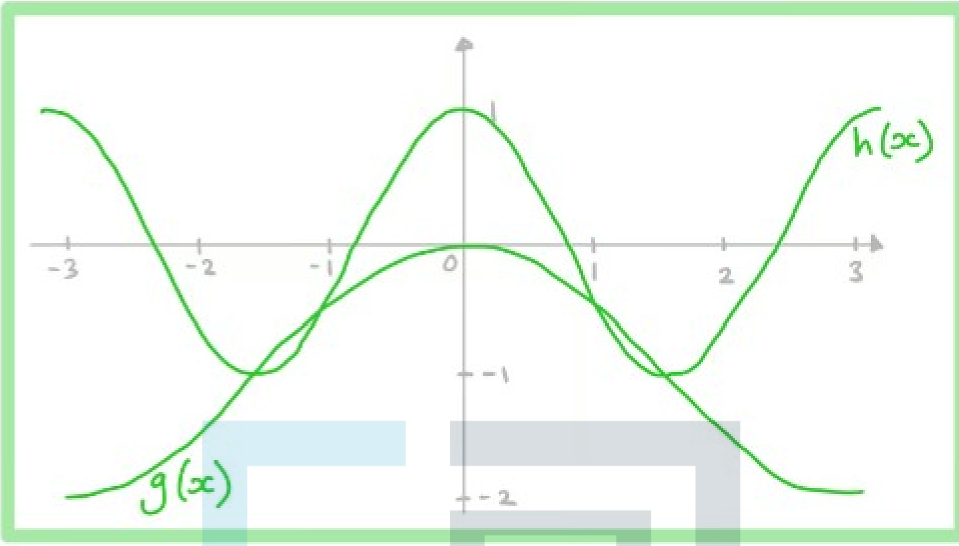
or $\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ or $\frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$

$\sin x = -1$: $x = \frac{3\pi}{2}$ or $\frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}$

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{7\pi}{2}}$$

Question 9 a) i) $g(x) = \cos(x) - 1$ $h(x) = \cos(2x)$

SHIFT DOWN BY 1 MAKE HALF AS WIDE



ii) g(x) TOUCHES x AXIS ONLY ONCE AT (0,0)

$\boxed{\text{1 ROOT}}$

b)

$$\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 1 - 2\sin^2 x \\ 2\cos^2 x - 1 \end{cases}$$

$$2\cos^2 x \cancel{-1} = \cos x \cancel{-1}$$

$$2\cos^2 x - \cos x = 0$$

$$\cos x\,(2\cos x - 1) = 0$$

$$\cos x = 0 \qquad 2\cos x - 1 = 0$$

$$\cos\theta = \tfrac{1}{2}$$

EXACT VALUES

$$\cos^{-1}(0) = \frac{\pi}{2} \qquad \cos^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{3}$$

$$-\pi \leq x \leq \pi$$

$$\boxed{x = -\frac{\pi}{2},\ -\frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{2}}$$

$$\frac{\pi}{2} \approx 1.5\ldots \qquad \frac{\pi}{3} \approx 1.0\ldots$$

LABEL ON GRAPH
USE PART (a) GRAPH TO HELP