DP IB Maths: AA HL

3.6 Trigonometric Equations & Identities

Contents

3.6.1 Simple Identities

Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of x or θ
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol ≡
  - This means 'identical to'

What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
    - This is the identity for $\tan\theta$
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$
  - $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H}$
  - $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$
  - $\frac{\sin\theta}{\cos\theta} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{O}{A} = \tan\theta$
- This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1

- $\tan\theta = \dfrac{O}{A} = \dfrac{\sin\theta}{\cos\theta}$
- The Pythagorean identity can be seen by considering a right-triangle with a hypotenuse of 1
  - Then (opposite)^2 + (adjacent)^2 = 1
  - Therefore $\sin^2\theta + \cos^2\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is $x^2 + y^2 = 1$
  - The coordinates on the unit circle are $(\cos\theta, \ \sin\theta)$
  - Therefore the equation of the unit circle could be written $\cos^2\theta + \sin^2\theta = 1$
- A third very useful identity is $\sin\theta = \cos(90° - \theta)$ or $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
  - This is not included in the formula booklet but is useful to remember

How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

Worked example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

$$2\sin^2 x - \cos x = 0$$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute : $2(1 - \cos^2 x) - \cos x = 0$

Expand : $2 - 2\cos^2 x - \cos x = 0$

Rearrange : $2\cos^2 x + \cos x - 2 = 0$

$$\boxed{a = 2, \quad b = 1, \quad c = -2}$$

3.6.2 Compound Angle Formulae

Compound Angle Formulae

What are the compound angle formulae?

- There are six **compound angle formulae** (also known as **addition formulae**), two each for **sin**, **cos** and **tan**:
- For **sin** the +/- sign on the left-hand side **matches** the one on the right-hand side
  - $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$
  - $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$
- For **cos** the +/- sign on the left-hand side is **opposite to** the one on the right-hand side
  - $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$
  - $\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$
- For **tan** the +/- sign on the left-hand side **matches** the one in the **numerator** on the right-hand side, and is **opposite to** the one in the **denominator**
  - $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
  - $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
- The compound angle formulae can all the found in the formula booklet, you do not need to remember them

When are the compound angle formulae used?

- The compound angle formulae are particularly useful when finding the values of trigonometric ratios without the use of a calculator
  - For example to find the value of sin15° rewrite it as sin (45 – 30)° and then
    - apply the compound formula for sin(A – B)
    - use your knowledge of exact values to calculate the answer
- The compound angle formulae are also used…
  - … to derive further multiple angle trig identities such as the double angle formulae
  - … in trigonometric proof
  - … to simplify complicated trigonometric equations before solving

How are the compound angle formulae for cosine proved?

- The proof for the compound angle identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$ can be seen by considering two coordinates on a unit circle, $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$
  - The angle between the positive x- axis and the point P is A
  - The angle between the positive x- axis and the point Q is B
  - The angle between P and Q is $B - A$

- Using the distance formula (Pythagoras) the distance PQ can be given as
  - $|PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$
- Using the cosine rule the distance PQ can be given as
  - $|PQ|^2 = 1^2 + 1^2 - 2(1)(1)\cos(B - A) = 2 - 2\cos(B - A)$
- Equating these two formulae, expanding and rearranging gives
  - $2 - 2\cos(B - A) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A \cos B - 2\sin A \sin B$
  - $2 - 2\cos(B - A) = 2 - 2(\cos A \cos B + \sin A \sin B)$
- Therefore $\cos(B - A) = \cos A \cos B + \sin A \sin B$
- Changing $-A$ for A in this identity and rearranging proves the identity for $\cos(A + B)$
  - $\cos(B - (-A)) = \cos(-A)\cos B + \sin(-A)\sin B = \cos A \cos B - \sin A \sin B$

How are the compound angle formulae for sine proved?

- The proof for the compound angle identity $\sin(A + B)$ can be seen by using the above proof for $\cos(B - A)$ and
  - Considering $\cos(\pi/2 - (A + B)) = \cos(\pi/2)\cos(A + B) + \sin(\pi/2)\sin(A + B)$
  - Therefore $\cos(\pi/2 - (A + B)) = \sin(A + B)$
  - Rewriting $\cos(\pi/2 - (A + B))$ as $\cos((\pi/2 - A) + B)$ gives
    - $\cos(\pi/2 - (A + B)) = \cos(\pi/2 - A)\cos B + \sin(\pi/2 - A)\sin B$
    - Using $\cos(\pi/2 - A) = \sin A$ and $\sin(\pi/2 - A) = \cos A$ and equating gives
      - $\sin(A + B) = \sin A \cos B + \cos A \cos B$
  - Substituting B for $-B$ proves the result for $\sin(A - B)$

How are the compound angle formulae for tan proved?

- The proof for the compound angle identities $\tan(A \pm B)$ can be seen by
  - Rewriting $\tan(A \pm B)$ as $\dfrac{\sin(A \pm B)}{\cos(A \pm B)}$
  - Substituting the compound angle formulae in
  - Dividing the numerator and denominator by $\cos A \cos B$

Worked example

a)

Show that $\tan\left(x+\frac{\pi}{4}\right) - \tan\left(x-\frac{\pi}{4}\right) = \frac{2(\tan^2 x+1)}{1-\tan^2 x}$

Use the compound angle formula for tan:

$$\tan\left(x+\frac{\pi}{4}\right) = \frac{\tan x + \tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}} = \frac{\tan x + 1}{1 - \tan x}$$

$$\tan\left(x-\frac{\pi}{4}\right) = \frac{\tan x - \tan\frac{\pi}{4}}{1+\tan x\tan\frac{\pi}{4}} = \frac{\tan x - 1}{1 + \tan x}$$

Put together and simplify:

$$\frac{\tan x+1}{1-\tan x} - \frac{\tan x - 1}{1+\tan x} = \frac{(\tan x+1)(1+\tan x) - (\tan x - 1)(1-\tan x)}{(1-\tan x)(1+\tan x)}$$

$$= \frac{\tan^2 x + 2\tan x + 1 - (-\tan^2 x + 2\tan x - 1)}{(1-\tan x)(1+\tan x)}$$

$$= \frac{2\tan^2 x + 2}{1-\tan^2 x}$$

$$\boxed{\frac{2(\tan^2 x + 1)}{1 - \tan^2 x}}$$

b)

Hence, solve $\tan\left(x+\frac{\pi}{4}\right) - \tan\left(x-\frac{\pi}{4}\right) = -4$ for $0 \le x \le \frac{\pi}{2}$

Use the answer found in (a) to write a new equation:

$$\frac{2(\tan^2 x + 1)}{1 - \tan^2 x} = -4$$

Rearrange and bring all terms in $\tan x$ to one side:

$$2(\tan^2 x + 1) = -4(1 - \tan^2 x)$$

$$2\tan^2 x + 2 = -4 + 4\tan^2 x$$

$$2\tan^2 x - 6 = 0$$

$$\tan^2 x = 3$$

$$\tan x = \pm\sqrt{3}$$

$$x = \frac{\pi}{3},\ -\frac{\pi}{3}$$ outside of given range

$$\boxed{x = \frac{\pi}{3}}$$

3.6.3 Double Angle Formulae

Double Angle Formulae

What are the double angle formulae?

- The **double angle formulae** for **sine** and **cosine** are:
  - $\sin 2\theta = 2\sin \theta\cos \theta$
  - $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$
  - $\tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$
- These can be found in the formula booklet
  - The formulae for sin and cos can be found in the SL section
  - The formula for tan can be found in the HL section

How are the double angle formulae derived?

- The double angle formulae can be derived from the compound angle formulae
- Simply replace *B* for *A* in each of the formulae and simplify
- For example
  - Sin 2*A* = sin (*A* + *A*) = sin*A*cos*A* + sin*A*cos*A* = 2sin*A*cos*A*

How are the double angle formulae used?

- Double angle formulae will often be used with…
  - ... trigonometry exact values
  - ... graphs of trigonometric functions
  - ... relationships between trigonometric ratios
- To help solve trigonometric equations which contain $\sin \theta\cos \theta$:
  - Substitute $\frac{1}{2}\sin 2\theta$ for $\sin \theta\cos \theta$
  - Solve for 2θ, finding all values in the range for 2θ
    - The range will need adapting for 2θ
  - Find the solutions for θ
- To help solve trigonometric equations which contain $\sin 2\theta$ and $\sin \theta$ or $\cos \theta$
  - Substitute $2\sin \theta\cos \theta$ for $\sin 2\theta$
  - Isolate all terms in θ
  - Factorise or use another identity to write the equation in a form which can be solved
- To help solve trigonometric equations which contain $\cos 2\theta$ and $\sin \theta$ or $\cos \theta$
  - Substitute either $2\cos^2 \theta - 1$ or $1 - 2\sin^2 \theta$ for $\cos 2\theta$
    - Choose the trigonometric ratio that is already in the equation
  - Isolate all terms in θ

- Solve
    - The equation will most likely be in the form of a quadratic
- To help solve trigonometric equations which contain tan 2θ
    - Substitute the double angle identity for tan 2θ
    - Rearrange, often this will lead to a quadratic equation in terms of tan θ
    - Solve
- Double angle formulae can be used in proving other trigonometric identities

Worked example

Without using a calculator, solve the equation $\sin 2\theta = \sin\theta$ for $0° \le \theta \le 360°$. Show all working clearly.

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$$2\sin\theta\cos\theta = \sin\theta$$

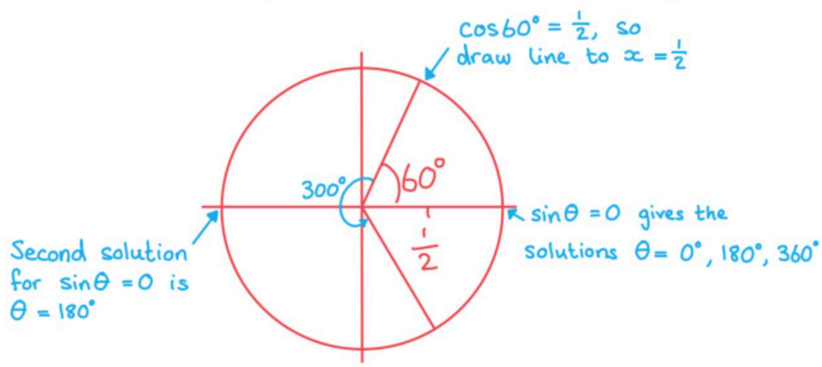
Bring both identities to one side:

$$2\sin\theta\cos\theta - \sin\theta = 0$$

Factorise: $\sin\theta(2\cos\theta - 1) = 0$

Find solutions: $\sin\theta = 0$, $\theta = 0$; $2\cos\theta - 1 = 0$, $\cos\theta = \frac{1}{2}$, $\theta = 60°$

Find secondary values within range:



$$\boxed{\theta = 0°,\ 60°,\ 180°,\ 300°,\ 360°}$$

3.6.4 Relationship Between Trigonometric Ratios

Relationship Between Trigonometric Ratios

What relationships between trigonometric ratios should I know?

- If you know a value for one trig ratio you can often use this to work out the value for the others without needing to find θ
- If you know that $\sin\theta = \frac{a}{b}$, where $a, b \in \mathbb{Z}^+$, you can:
  - Sketch a right-triangle with *a* opposite θ and *b* on the hypotenuse
  - Use Pythagoras' theorem to find the value of the adjacent side
  - Use SOHCAHTOA to find the values of cos θ and tan θ
- If you know a value for sin θ or cos θ you can use the Pythagorean relationship
  - $\sin^2\theta + \cos^2\theta = 1$
  - to find the value of the other
- If you know a value for sin θ or cos θ you can use the double angle formulae to find the value of sin 2θ or cos 2θ
- If you know a value for tan θ you can use the double angle formulae to find the value of tan 2θ
- If you know two out of the three values for sin θ, cos θ or tan θ you can use the identity in tan
  - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
  - to find the value of the third ratio

How do we determine whether a trigonometric ratio will be positive or negative?

- It is possible to determine whether a trigonometric ratio will be positive or negative by looking at the size of the angle and considering the **unit circle**
  - Angles in the range 0° < θ° < 90° will be positive for all three ratios
  - Angles in the range 90° < θ° < 180° will be positive for sin and negative for cos and tan
  - Angles in the range 180° < θ° < 270° will be positive for tan and negative for sin and cos
  - Angles in the range 270° < θ° < 360° will be positive for cos and negative for sin and tan
- The ratios for angles of 0°, 90°, 180°, 270° and 360° are either 0, 1, -1 or undefined
  - You should know these ratios or know how to derive them without a calculator

Worked example

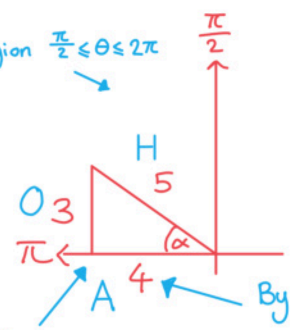
The value of $\sin\alpha = \dfrac{3}{5}$ for $\dfrac{\pi}{2} \le \alpha \le \pi$. Find:

i) $\cos\alpha$

Method 1: Use right-triangle:

$\frac{\pi}{2} \le \alpha \le \pi$

Answers in region $\frac{\pi}{2} \le \theta \le 2\pi$



$\sin\alpha = \frac{3}{5}$

$\sin\alpha = \frac{O}{H}$

By Pythagoras' theorem:

$c = \sqrt{5^2 - 3^2} = 4$

x coordinate of -4 gives solution in region

$\cos\alpha = \frac{A}{H} = -\frac{4}{5}$

$$\boxed{\cos\alpha = -\frac{4}{5}}$$

Method 2: Use Pythagorean identity:

$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\frac{3}{5}\right)^2$

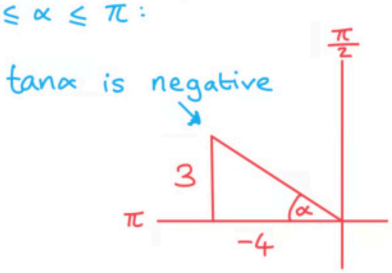
$\cos\alpha = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

Check which solution is in range.

ii) $\tan\alpha$

Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$

Check if $\tan\alpha = -\frac{3}{4}$ is in the correct range for $\frac{\pi}{2} \le \alpha \le \pi$:



$$\boxed{\tan\alpha = -\frac{3}{4}}$$

iii) $\sin 2\alpha$

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$$
$$= \frac{-24}{25}$$

$$\boxed{\sin\alpha = \frac{3}{5} \quad \cos\alpha = \frac{-4}{5}}$$

$$\boxed{\sin 2\alpha = -\frac{24}{25}}$$

iv) $\cos 2\alpha$

Double angle identity : $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$\cos 2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\boxed{\cos 2\alpha = \frac{7}{25}}$$

v) $\tan 2\alpha$

Using identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

$$\boxed{\tan 2\alpha = -\frac{24}{7}}$$

3.6.5 Linear Trigonometric Equations

Trigonometric Equations: sinx = k

How are trigonometric equations solved?

- Trigonometric equations can have an infinite number of solutions
  - For an equation in sin or cos you can add 360° or 2π to each solution to find more solutions
  - For an equation in tan you can add 180° or π to each solution
- When solving a trigonometric equation you will be given a range of values within which you should find all the values
- Solving the equation normally and using the inverse function on your calculator or your knowledge of **exact values** will give you the **primary value**
- The **secondary values** can be found with the help of:
  - The **unit circle**
  - The **graphs of trigonometric functions**

How are trigonometric equations of the form sin x = k solved?

- It is a good idea to sketch the graph of the trigonometric function first
  - Use the given range of values as the domain for your graph
  - The intersections of the graph of the function and the line y = k will show you
    - The location of the solutions
    - The number of solutions
  - You will be able to use the symmetry properties of the graph to find all secondary values within the given range of values
- The method for finding secondary values are:
  - For the equation $\sin x = k$ the primary value is $x_1 = \sin^{-1} k$
    - A secondary value is $x_2 = 180° - \sin^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\cos x = k$ the primary value is $x_1 = \cos^{-1} k$
    - A secondary value is $x_2 = -\cos^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\tan x = k$ the primary value is $x = \tan^{-1} k$
    - All secondary values within the range can be found using $x \pm 180n$ where $n \in \mathbb{N}$

Worked example

Solve the equation $2\cos x = -1$, finding all solutions in the range $-\pi \le x \le \pi$.

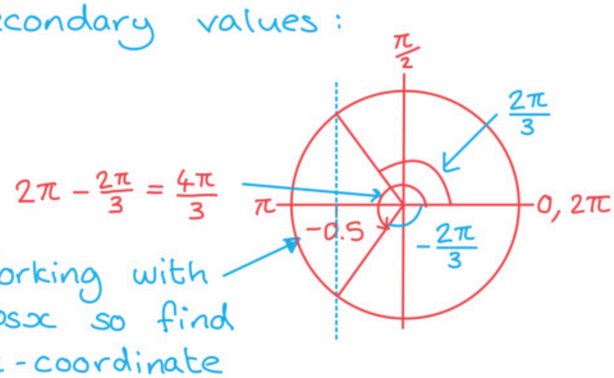
Isolate $\cos x$: $\cos x = \frac{-1}{2}$

$x = \cos^{-1}\left(-\frac{1}{2}\right)$

Use GDC or knowledge of exact values

$= \frac{2\pi}{3}$ ← Primary value

Find secondary values :



$\frac{2\pi}{3} \pm 2\pi n$ and $\frac{4\pi}{3} \pm 2\pi n$

Find all answers in range $-\pi \le x \le 3\pi$

$$\boxed{-\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}}$$

Trigonometric Equations: sin(ax + b) = k

How can I solve equations with transformations of trig functions?

- Trigonometric equations in the form $\sin(ax + b)$ can be solved in more than one way
- The easiest method is to consider the transformation of the angle as a substitution
  - For example let $u = ax + b$
- Transform the given interval for the solutions in the same way as the angle
  - For example if the given interval is $0° \le x \le 360°$ the new interval will be
  - $(a(0°) + b) \le u \le (a(360°) + b)$
- Solve the function to find the primary value for u
- Use either the unit circle or sketch the graph to find all the other solutions in the range for u
- Undo the substitution to convert all of the solutions back into the corresponding solutions for x
- Another method would be to sketch the transformation of the function
  - If you use this method then you will not need to use a substitution for the range of values

Worked example

Solve the equation $2\cos(2x - 30°) = -1$, finding all solutions in the range $-360° \le x \le 360°$.

$2\cos(2x - 30°) = -1 \qquad -360° \le x \le 360°$

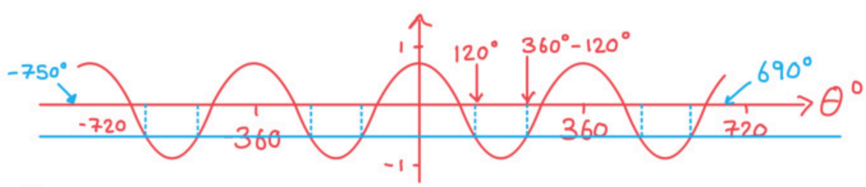
Start by changing the range : $-750° \le 2x - 30 \le 690°$

Substitute $\theta = 2x - 30$:

$2\cos\theta = -1 \qquad -750° \le \theta \le 690°$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$ ← Primary value



From the sketch you can see there are 8 solutions :

$\theta = 120° \pm 360°$ and $\theta = 240° \pm 360°$

$\theta = -600°, -480°, -240°, -120°, 120°, 240°, 480°, 600°$

Solve for x : $x = \dfrac{\theta° + 30}{2}$

$$\boxed{x = -285°, -225°, -105°, -45°, 75°, 135°, 255°, 315°}$$

3.6.6 Quadratic Trigonometric Equations

Quadratic Trigonometric Equations

How are quadratic trigonometric equations solved?

- A quadratic trigonometric equation is one that includes either $\sin^2 \theta$, $\cos^2 \theta$ or $\tan^2 \theta$
- Often the **identity** $\sin^2 \theta + \cos^2 \theta = 1$ can be used to rearrange the equation into a form that is possible to solve
  - If the equation involves both sine and cosine then the **Pythagorean identity** should be used to write the equation in terms of just one of these functions
- Solve the **quadratic equation** using your GDC, the quadratic equation or factorisation
  - This can be made easier by changing the function to a single letter
    - Such as changing $2\cos^2 \theta - 3\cos \theta - 1 = 0$ to $2c^2 - 3c - 1 = 0$
- A quadratic can give up to two solutions
  - You must consider both solutions to see whether a real value exists
  - Remember that solutions for $\sin \theta = k$ and $\cos \theta = k$ only exist for $-1 \leq k \leq 1$
  - Solutions for $\tan \theta = k$ exist for all values of k
- Find all solutions within the given interval
  - There will often be more than two solutions for one quadratic equation
  - The best way to check the number of solutions is to sketch the graph of the function

Worked example

Solve the equation $11\sin x - 7 = 5\cos^2 x$, finding all solutions in the range $0 \le x \le 2\pi$.

Use the identity $\cos^2 x = 1 - \sin^2 x$ to write equation in terms of $\sin x$:

$$11\sin x - 7 = 5(1 - \sin^2 x)$$ in formula booklet.

$$= 5 - 5\sin^2 x$$

Move all terms to one side:

$$11\sin x - 7 - (5 - 5\sin^2 x) = 0$$

Spot the hidden quadratic:

$$11\sin x - 7 - 5 + 5\sin^2 x = 0$$

$$5\sin^2 x + 11\sin x - 12 = 0$$

$$\sin x = \frac{4}{5} \quad \text{or} \quad \sin x = -3$$

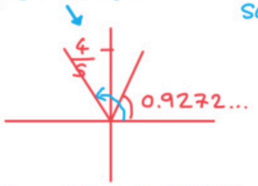
no solution so disregard.

$$\sin x = \frac{4}{5}$$

$$x = 0.9272\ldots$$

primary solution

sinx so use y coordinate.



$$x = \pi - 0.9272\ldots$$

$$= 2.214\ldots$$

secondary solution

$$\boxed{x = 0.927,\ 2.21 \ (3\text{ s.f.})}$$