DP IB Maths: AA SL

3.6 Trigonometric Equations & Identities

Contents

3.6.1 Simple Identities

Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of x or θ
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol $\equiv$
  - This means 'identical to'

What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - This is the identity for $\tan\theta$
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$
  - $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{O}{H}}{\frac{A}{H}} = \dfrac{O}{A} = \tan\theta$
- This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1

  - $\tan\theta = \dfrac{O}{A} = \dfrac{\sin\theta}{\cos\theta}$
- The Pythagorean identity can be seen by considering a right-triangle with a hypotenuse of 1
  - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore $\sin^2\theta + \cos^2\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is $x^2 + y^2 = 1$
  - The coordinates on the unit circle are $(\cos\theta,\ \sin\theta)$
  - Therefore the equation of the unit circle could be written $\cos^2\theta + \sin^2\theta = 1$
- A third very useful identity is $\sin\theta = \cos(90° - \theta)$ or $\sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$
  - This is not included in the formula booklet but is useful to remember

How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

Worked example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

$2\sin^2 x - \cos x = 0$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute : $2(1 - \cos^2 x) - \cos x = 0$

Expand : $2 - 2\cos^2 x - \cos x = 0$

Rearrange : $2\cos^2 x + \cos x - 2 = 0$

$\boxed{a = 2, \ b = 1, \ c = -2}$

3.6.2 Double Angle Formulae

Double Angle Formulae

What are the double angle formulae?

- The **double angle formulae** for **sine** and **cosine** are:
  - $\sin 2\theta = 2\sin\theta\cos\theta$
  - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- These can be found in the formula booklet
  - You do not need to know how to derive them

How are the double angle formulae used?

- Double angle formulae will often be used with…
  - … trigonometry exact values
  - … graphs of trigonometric functions
  - … relationships between trigonometric ratios
- To help solve trigonometric equations which contain $\sin\theta\cos\theta$:
  - Substitute $\frac{1}{2}\sin 2\theta$ for $\sin\theta\cos\theta$
  - Solve for 2θ, finding all values in the range for 2θ
    - The range will need adapting for 2θ
  - Find the solutions for θ
- To help solve trigonometric equations which contain $\sin 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute $2\sin\theta\cos\theta$ for $\sin 2\theta$
  - Isolate all terms in θ
  - Factorise or use another identity to write the equation in a form which can be solved
- To help solve trigonometric equations which contain $\cos 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute either $2\cos^2\theta - 1$ or $1 - 2\sin^2\theta$ for $\cos 2\theta$
    - Choose the trigonometric ratio that is already in the equation
  - Isolate all terms in θ
  - Solve
    - The equation will most likely be in the form of a quadratic
- Double angle formulae can be used in proving other trigonometric identities

Worked example

Without using a calculator, solve the equation $\sin 2\theta = \sin \theta$ for $0° \leq \theta \leq 360°$. Show all working clearly.

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$$2\sin\theta\cos\theta = \sin\theta$$

Bring both identities to one side:

$$2\sin\theta\cos\theta - \sin\theta = 0$$

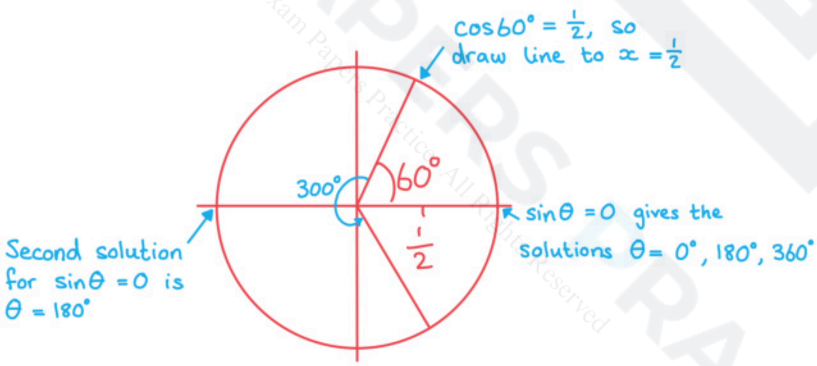
Factorise: $\sin\theta(2\cos\theta - 1) = 0$

Find solutions: $\sin\theta = 0$ $\qquad 2\cos\theta - 1 = 0$

$\theta = 0$ $\qquad \cos\theta = \frac{1}{2}$

$\theta = 60°$

Find secondary values within range:



$$\boxed{\theta = 0°, 60°, 180°, 300°, 360°}$$

3.6.3 Relationship Between Trigonometric Ratios

Relationship Between Trigonometric Ratios

What relationships between trigonometric ratios should I know?

- If you know a value for one trig ratio you can often use this to work out the value for the others without needing to find θ
- If you know that $\sin\,\theta = \dfrac{a}{b}$, where $a,\ b\ \in\ \mathbb{Z}^+$, you can:
  - Sketch a right-triangle with a opposite θ and b on the hypotenuse
  - Use Pythagoras' theorem to find the value of the adjacent side
  - Use SOHCAHTOA to find the values of cos θ and tan θ
- If you know a value for sin θ or cos θ you can use the Pythagorean relationship
  - $\sin^2\,\theta + \ \cos^2\,\theta = 1$
  - to find the value of the other
- If you know a value for sin θ or cos θ you can use the double angle formulae to find the value of sin 2θ or cos 2θ
- If you know two out of the three values for sin θ, cos θ or tan θ you can use the identity in tan
  - $\tan\,\theta = \dfrac{\sin\,\theta}{\cos\,\theta}$
  - to find the value of the third ratio

How do we determine whether a trigonometric ratio will be positive or negative?

- It is possible to determine whether a trigonometric ratio will be positive or negative by looking at the size of the angle and considering the **unit circle**
  - Angles in the range $0° < \theta° < 90°$ will be positive for all three ratios
  - Angles in the range $90° < \theta° < 180°$ will be positive for sin and negative for cos and tan
  - Angles in the range $180° < \theta° < 270°$ will be positive for tan and negative for sin and cos
  - Angles in the range $270° < \theta° < 360°$ will be positive for cos and negative for sin and tan
- The ratios for angles of 0°, 90°, 180°, 270° and 360° are either 0, 1, -1 or undefined
  - You should know these ratios or know how to derive them without a calculator

Worked example

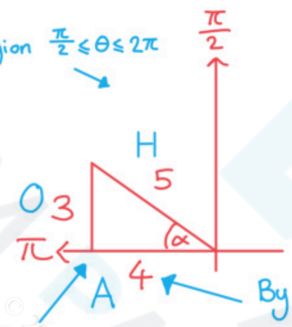
The value of $\sin\alpha = \frac{3}{5}$ for $\frac{\pi}{2} \leq \alpha \leq \pi$. Find:

i) $\cos\alpha$

Method 1: Use right-triangle:

$\frac{\pi}{2} \leq \alpha \leq \pi$

Answers in region $\frac{\pi}{2} \leq \theta \leq 2\pi$



$\sin\alpha = \frac{3}{5}$

$\sin\alpha = \frac{O}{H}$

By Pythagoras' theorem:

$c = \sqrt{5^2 - 3^2} = 4$

x coordinate of -4 gives solution in region

$\cos\alpha = \frac{A}{H} = -\frac{4}{5}$

$$\boxed{\cos\alpha = -\frac{4}{5}}$$

Method 2: Use Pythagorean identity:

$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\frac{3}{5}\right)^2$

$\cos\alpha = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

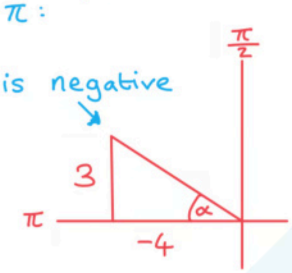
Check which solution is in range.

ii) $\tan\alpha$

Use $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\frac{3}{5}}{-\frac{4}{5}} = -\dfrac{3}{4}$

Check if $\tan\alpha = -\frac{3}{4}$ is in the correct range for $\frac{\pi}{2} \le \alpha \le \pi$:

$\tan\alpha$ is negative



$$\boxed{\tan\alpha = -\frac{3}{4}}$$

iii) $\sin 2\alpha$

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$

$= -\frac{24}{25}$

$$\boxed{\sin\alpha = \frac{3}{5} \quad \cos\alpha = -\frac{4}{5}}$$

$$\boxed{\sin 2\alpha = -\frac{24}{25}}$$

iv) $\cos 2\alpha$

Double angle identity : $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$\cos 2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\boxed{\cos 2\alpha = \frac{7}{25}}$$

v) $\tan 2\alpha$

Using identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

$$\boxed{\tan 2\alpha = -\frac{24}{7}}$$

3.6.4 Linear Trigonometric Equations

Trigonometric Equations: sinx = k

How are trigonometric equations solved?

- Trigonometric equations can have an infinite number of solutions
  - For an equation in sin or cos you can add 360° or 2π to each solution to find more solutions
  - For an equation in tan you can add 180° or π to each solution
- When solving a trigonometric equation you will be given a range of values within which you should find all the values
- Solving the equation normally and using the inverse function on your calculator or your knowledge of **exact values** will give you the **primary value**
- The **secondary values** can be found with the help of:
  - The **unit circle**
  - The **graphs of trigonometric functions**

How are trigonometric equations of the form sin x = k solved?

- It is a good idea to sketch the graph of the trigonometric function first
  - Use the given range of values as the domain for your graph
  - The intersections of the graph of the function and the line y = k will show you
    - The location of the solutions
    - The number of solutions
  - You will be able to use the symmetry properties of the graph to find all secondary values within the given range of values
- The method for finding secondary values are:
  - For the equation $\sin x = k$ the primary value is $x_1 = \sin^{-1} k$
    - A secondary value is $x_2 = 180° - \sin^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\cos x = k$ the primary value is $x_1 = \cos^{-1} k$
    - A secondary value is $x_2 = -\cos^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\tan x = k$ the primary value is $x = \tan^{-1} k$
    - All secondary values within the range can be found using $x \pm 180n$ where $n \in \mathbb{N}$

Worked example

Solve the equation $2\cos x = -1$, finding all solutions in the range $-\pi \le x \le \pi$.

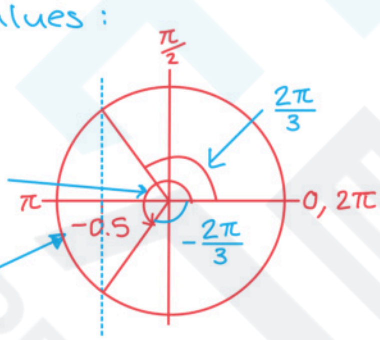
Isolate $\cos x$: $\cos x = \frac{-1}{2}$

$x = \cos^{-1}\left(-\frac{1}{2}\right)$

Use GDC or knowledge of exact values $= \frac{2\pi}{3}$ ← Primary value

Find secondary values :

$2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$



Working with $\cos x$ so find x-coordinate

$\frac{2\pi}{3} \pm 2\pi n$ and $\frac{4\pi}{3} \pm 2\pi n$

Find all answers in range $-\pi \le x \le 3\pi$

$$\boxed{-\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}}$$

Trigonometric Equations: sin(ax + b) = k

How can I solve equations with transformations of trig functions?

- Trigonometric equations in the form $\sin(ax + b)$ can be solved in more than one way
- The easiest method is to consider the transformation of the angle as a substitution
  - For example let $u = ax + b$
- Transform the given interval for the solutions in the same way as the angle
  - For example if the given interval is $0° \le x \le 360°$ the new interval will be
  - $(a(0°) + b) \le u \le (a(360°) + b)$
- Solve the function to find the primary value for u
- Use either the unit circle or sketch the graph to find all the other solutions in the range for u
- Undo the substitution to convert all of the solutions back into the corresponding solutions for x
- Another method would be to sketch the transformation of the function
  - If you use this method then you will not need to use a substitution for the range of values

Worked example

Solve the equation $2\cos(2x - 30°) = -1$, finding all solutions in the range $-360° \le x \le 360°$.

$2\cos(2x - 30°) = -1 \qquad -360° \le x \le 360°$

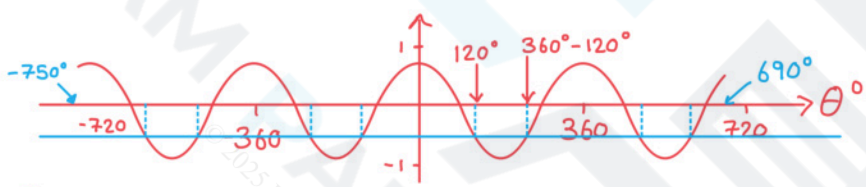
Start by changing the range : $-750° \le 2x - 30 \le 690°$

Substitute $\theta = 2x - 30$:

$2\cos\theta = -1 \qquad -750° \le \theta \le 690°$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120° \leftarrow$ primary value



From the sketch you can see there are 8 solutions :

$\theta = 120° \pm 360°$ and $\theta = 240° \pm 360°$

$\theta = -600°, -480°, -240°, -120°, 120°, 240°, 480°, 600°$

Solve for x : $x = \dfrac{\theta° + 30}{2}$

$$\boxed{x = -285°, -225°, -105°, -45°, 75°, 135°, 255°, 315°}$$

3.6.5 Quadratic Trigonometric Equations

Quadratic Trigonometric Equations

How are quadratic trigonometric equations solved?

- A quadratic trigonometric equation is one that includes either $\sin^2\theta$, $\cos^2\theta$ or $\tan^2\theta$
- Often the **identity** $\sin^2\theta + \cos^2\theta = 1$ can be used to rearrange the equation into a form that is possible to solve
  - If the equation involves both sine and cosine then the **Pythagorean identity** should be used to write the equation in terms of just one of these functions
- Solve the **quadratic equation** using your GDC, the quadratic equation or factorisation
  - This can be made easier by changing the function to a single letter
    - Such as changing $2\cos^2\theta - 3\cos\theta - 1 = 0$ to $2c^2 - 3c - 1 = 0$
- A quadratic can give up to two solutions
  - You must consider both solutions to see whether a real value exists
  - Remember that solutions for $\sin\theta = k$ and $\cos\theta = k$ only exist for $-1 \le k \le 1$
  - Solutions for $\tan\theta = k$ exist for all values of k
- Find all solutions within the given interval
  - There will often be more than two solutions for one quadratic equation
  - The best way to check the number of solutions is to sketch the graph of the function

Worked example

Solve the equation $11\sin x - 7 = 5\cos^2 x$, finding all solutions in the range $0 \leq x \leq 2\pi$.

Use the identity $\cos^2 x = 1 - \sin^2 x$ to write equation in terms of $\sin x$:

$$11\sin x - 7 = 5(1 - \sin^2 x)$$

in formula booklet.

$$= 5 - 5\sin^2 x$$

Move all terms to one side:

$$11\sin x - 7 - (5 - 5\sin^2 x) = 0$$

Spot the hidden quadratic:

$$11\sin x - 7 - 5 + 5\sin^2 x = 0$$

$$5\sin^2 x + 11\sin x - 12 = 0$$

$$\sin x = \frac{4}{5} \quad \text{or} \quad \sin x = -3$$

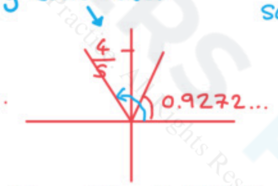
no solution so disregard.

$$\sin x = \frac{4}{5}$$

$$x = 0.9272\ldots$$

primary solution

sinx so use y coordinate.



$$x = \pi - 0.9272\ldots$$

$$= 2.214\ldots$$

secondary solution

$$\boxed{x = 0.927,\ 2.21 \ (3\text{ s.f.})}$$